

# CHAOS IN OCEAN PHYSICS

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## ABSTRACT

Three topics relating to chaotic ocean physics are discussed. These are (1) low order El Niño dynamics, (2) lateral stirring processes, and (3) linear ocean waves in the geometric limit. Each topic is discussed separately; emphasis, in each case, is given to the manner in which ideas associated with chaos and low-order dynamical systems complement more traditional approaches to the same problem.

## INTRODUCTION

In this paper three topics relating to chaotic ocean physics are discussed. This list is not intended to be an exhaustive list of topics in ocean physics to which ideas relating to chaos can be applied. Our discussion of these three topics—which were chosen because the author has some familiarity with them—serves to illustrate several important concepts likely to be useful in other oceanographic applications as well. It is our feeling that the ideas relating to chaotic dynamical systems discussed in this paper are useful but must be applied in a sober fashion which complements more traditional approaches. When properly applied, these ideas provide a vehicle to increase our understanding of various physical processes in the ocean in an evolutionary fashion. Expectations of gaining new insight of a revolutionary nature are not likely to be realized.

In each of the three sections that follow, we discuss a topic in ocean physics (low-order El Niño dynamics, lateral stirring processes, linear ocean waves in the geometric limit) to which ideas associated with chaos can be applied. Background material relating to dynamical systems and chaotic dynamics is introduced as necessary in the context of the problems treated. This approach is natural inasmuch as our intention is not to provide a tutorial on chaos; instead, we seek to demonstrate that these ideas are useful in the context of specific problems in ocean physics. All three topics discussed in this paper are treated in more detail elsewhere; references are provided below. So as not to duplicate this material, we focus here on the rationale for applying ideas relating to chaos and low-order dynamical systems. Stated somewhat differently, in this paper we focus more on the questions being addressed than on details of the subsequent analysis. Some unifying comments and observations concerning chaotic ocean physics are included in the final section.

## LOW-ORDER EL NIÑO DYNAMICS

The El Niño/Southern Oscillation (ENSO) system is a quasi-periodic oscillation of the tropical Pacific Ocean and overlying atmosphere (see, e.g., Enfield, 1989). The ENSO system involves interactions among eastern basin sea surface temperature (SST), zonal trade winds and the thermocline depth (Bjerknes, 1969; Wyrski, 1975). El Niño events—characterized by anomalously high eastern basin SST, weak trade winds, and a shallow western basin thermocline—are separated by three to five years, typically.

Models of the ENSO system vary considerably in complexity. At one extreme are coupled ocean-atmosphere general circulation models (see, e.g., Neelin, 1990). That such models produce ENSO-like behavior should come as no surprise; ENSO behavior is surely contained in the complicated coupled equations of motion/state which were numerically solved. It is our feeling that simpler models—provided they adequately reproduce essential features of the system being modeled—are more insightful inasmuch as they better elucidate the essential physical processes involved. This leads naturally to the question of whether the essential physics of the ENSO system can be captured in simpler models.

The simplest type of model of the ENSO system which has been proposed consists of a small number ( $n$ , say) of autonomous ordinary differential equations,

$$\frac{dx}{dt} = f(x). \quad (1)$$

The solution  $\underline{x}(t)$  of these equations describes the temporal evolution of the system. The  $x_i$ 's ( $i = 1, 2, \dots, n$ ) in such a model would include variables such as anomalies of eastern basin SST, zonal winds and western basin thermocline depth. Vallis (1986) was the first to propose a model of the ENSO system of this type. That this model produces unphysical behavior for some choices of parameters (see, e.g., Vallis, 1988) is, in our opinion, not terribly important: the significance of the Vallis (1986) paper is the suggestion that the essential physics of the ENSO system can be captured in severely truncated physical model consisting of a low-order, autonomous dynamical system. The word autonomous means that the function  $f$  in (1) does not depend explicitly on time; physically, this restriction means that any quasi-oscillatory behavior in  $x_i(t)$ —which might be associated with the occurrence of El Niño events—is the result of internal, self-sustained dynamical processes rather than being the response to external stochastic forcing. More recently, improved low-order models (autonomous dynamical systems) of the ENSO system have been proposed by Schopf and Suarez (1988) and Münnich et al. (1991). Before proceeding, it is worth noting that the notion of simple ENSO dynamics—during the growth phase of El Niño events, at least—is generally accepted and dates back to the seminal work of Bjerknes (1969) and Wyrski (1975); the notion that the complete ENSO cycle—and, in particular, the triggering of El Niño events—results from

internal dynamics (i.e., that these oscillations are self-sustained) is not universally accepted.

These considerations led Bauer and Brown (1992) to address the question of whether *observations* of the ENSO system are consistent with underlying low-order dynamics. This question was addressed via the process of phase space reconstruction whereby discrete samples of a single variable,  $y(t_j)$ ,  $j = 1, 2, \dots$  (monthly samples of eastern basin SST were used in the Bauer and Brown analysis), are used to construct a discretely sampled multidimensional phase space portrait,  $y(t_k)$ ,  $k = 1, 2, \dots$ . A simple way to carry out this process is to use delay time coordinates:  $y_1(t_k) = y(t_k)$ ,  $y_2(t_k) = y(t_{k+1})$ ,  $y_3(t_k) = y(t_{k+2})$ , etc. Surprisingly, perhaps, the reconstructed, discretely sampled phase space trajectory  $\underline{y}(t_k)$  constructed in this fashion can be shown (Broomhead and King, 1986), under appropriate conditions, to reproduce with only minor distortion (a diffeomorphism) the true multidimensional phase space portrait  $\underline{x}(t)$  of the underlying dynamical system. Unfortunately, this procedure is sensitive to noise and therefore generally works poorly on geophysical data. The shortcoming was overcome by Bauer and Brown by using a technique developed by Broomhead and King (1986)—see also Vautard and Ghil (1989)—wherein temporal empirical orthogonal functions are used as basis functions for the reconstructed phase space trajectory. Details of this analysis will not be repeated here. The results of this analysis suggest that the underlying ENSO dynamics are approximately those of a low-order system; we urge the reader to carefully assess the evidence presented and come to his/her own conclusion.

It is worth emphasizing in this context that the question of chaotic ENSO dynamics is secondary to the question of whether ENSO dynamics are approximately those of a low-order system. If the later question is answered affirmatively, then questions concerning chaotic behavior become relevant. Among these are (1) Does the system evolve chaotically, and if so, what is the predictability timescale (reciprocal of the largest positive Lyapunov exponent)? (2) What is the dimension of the corresponding attractor? At the present time these questions are, in our opinion, premature. It is worth pointing out, however, that if the underlying dynamics are approximately those of a low-order system—even a chaotic one—this would lead to some long-term predictability in the sense that it would be known that the system's state vector  $\underline{x}$  must, at all times, lie on some attractor—although its precise position may not be predictable.

## LATERAL OCEAN STIRRING PROCESSES

In the ocean, many water properties such as temperature, salinity, oxygen content or pollutant concentration can be treated approximately as passive fluid parcel markers. Passive means that the flow field evolves independently of the initial distribution of the tracer. In order to understand the distribution of these oceanic tracers and how they evolve in time, one needs to understand the process by which passive tracers get redistributed. Our discussion of this process focuses on the lateral stirring (advective

tracer transport) process; we ignore the quasi-diffusive 3-d behavior that takes place at the smallest scales (internal wave and smaller).

The advective transport of a passive tracer in a two dimensional incompressible flow is described by the equation,

$$\frac{\partial \theta}{\partial t} - \frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} = 0 \quad (2)$$

subject to the initial condition  $\theta(x, y, 0) = \theta_0(x, y, t)$ . Here  $\theta(x, y, t)$  is the tracer concentration and  $\psi(x, y, t)$  is the streamfunction. It follows from (2) that  $\theta$  is constant following particle trajectories,  $x(t)$ ,  $y(t)$ , which satisfy

$$\frac{dx}{dt} = -\frac{\partial \psi}{\partial y}, \quad \frac{dy}{dt} = \frac{\partial \psi}{\partial x} \quad (3)$$

Thus, in order to understand the temporal evolution of  $\theta(x, y, t)$ —even in a statistical sense—one needs to understand the behavior of particle trajectories and understand the implications of the form of the Lagrangian equations of motion (3).

The Lagrangian equations of motion constitute a generally nonautonomous Hamiltonian system with one degree of freedom;  $\psi(x, y, t)$  plays the role of the Hamiltonian  $H(p, q, t)$ . It is extremely important to distinguish integrable Hamiltonian systems from nonintegrable ones. For the system (3) integrability implies that there exists a single-valued function  $\chi(x, y, t)$  which is constant following particle trajectories,  $d\chi/dt = 0$ . If the flow is steady,  $\partial\psi/\partial t = 0$ , then the system of equations is said to be autonomous and the streamfunction is the required constant of the motion,  $d\psi/dt = 0$ . This follows from equations (3). In nonsteady flows, however, the equations of motion (3) are nonautonomous and are generally nonintegrable. This observation is important inasmuch as nonintegrability is a necessary—but not sufficient—condition for chaotic motion (see, e.g., Tabor, 1989).

The distinction between chaotic and nonchaotic particle trajectories is extremely important in the context of passive tracer transport. The reason is that chaotic particle trajectories exhibit extreme sensitivity to their initial conditions. This means that neighboring particle trajectories diverge from one another at an exponential rate, on average. It follows that material lines of fluid will also grow exponentially, on average. This type of behavior leads to very efficient stirring (advective transport) of a tracer, and, in turn, enhances the mixing (diffusive transport) of the tracer at smaller scales. These ideas are discussed in more detail by Ottino (1990) (see also the contribution by Ottino in this volume) and Brown and Smith (1990, 1991). The latter publications also address the question of whether proxy ocean particle trajectories (acoustically tracked submerged SOFAR floats) exhibit extreme sensitivity. Previously, Osborne et al. (1986) had addressed this question

using satellite-tracked surface drifters. This work suggests that float/drifter trajectories do exhibit the important property of extreme sensitivity which is associated with chaotic systems.

It is important to note, however, that typical oceanographic realizations of  $\psi(x,y,t)$  are significantly more complicated than the idealized systems to which notions relating to chaos are normally associated. Specifically, almost integrable systems with periodic time-dependence are fairly well understood (see, e.g., Tabor, 1989). In such systems, the onset of chaos is associated with resonances between periodic motion in the nearby integrable system and the period of the temporal variations of the streamfunction. It is not clear whether results which apply to time-periodic streamfunctions carry over to the problem where the streamfunction has more general time dependence; there remains a significant gap between the complexity of the ocean and that of the idealized systems treated in textbooks on nonlinear dynamics.

This gap in complexity offers challenges to both oceanographers and nonlinear dynamicists and provides the opportunity for the two groups work together in a mutually beneficial fashion. In fact, this has already happened. In the aforementioned work of Osborne et al. (1986), the authors argued that the fractal characteristics of drifter trajectories was attributable to underlying stochasticity (power law energy spectrum of the velocity field) rather than being associated with a strange attractor. This work led to several studies on the relationship between stochasticity and fractal behavior.

## LINEAR OCEAN WAVES IN THE GEOMETRIC LIMIT

In the geometric (ray theoretical) limit, any type of linear wave motion can be described using a ray approximation (see, e.g., Lighthill, 1978). Such a description is valid when the properties of the ocean, including its boundaries, vary slowly on a scale of wavelengths. The ray equations are

$$\frac{dx_i}{dt} = \frac{\partial \omega}{\partial k_i}, \quad \frac{dk_i}{dt} = -\frac{\partial \omega}{\partial x_i} \quad (4)$$

where

$$\omega = \omega(\underline{k}, \underline{x}). \quad (5)$$

Here the  $x_i$ 's are position coordinates and the  $k_i$ 's are the corresponding components of the wavenumber vector. The form of the function  $\omega(\underline{k}, \underline{x})$ —the dispersion relation—depends on the type of wave being considered. For example, for surface gravity waves propagating in water of variable depth  $h(\underline{x}) = h(x,y)$ ,

$$\omega(\underline{k}, \underline{x}) = [g|\underline{k}| \tanh(|\underline{k}|h(x))]^{1/2} \quad (6)$$

In the following, some important ideas are illustrated using this form of the dispersion relation. We emphasize, however, that equations (4) and (5) are very general and that the following considerations are applicable to any type of linear wave motion.

Equations (4) and (6) constitute an autonomous Hamiltonian system with two degrees of freedom;  $\omega(\underline{k}, \underline{x})$  serves as the Hamiltonian  $H(\underline{p}, \underline{q})$ . Autonomous means that the Hamiltonian function does not depend explicitly on the dependent variable, time. Integrability of such a system requires that two independent constants of the motion exist. One of these is  $\omega(\underline{k}, \underline{x})$ ; it follows from equations (4) that  $d\omega/dt = 0$ . Only for very special bathymetric variations  $h(x,y)$  does the second required constant of the motion exist. For example, if  $h = h(x)$ , then it follows from the second of equations (4) that  $dk_y/dt = 0$ ; under such conditions  $k_y$  is a second constant of the motion. Such behavior is not typical, however.

In the absence of a second constant of the motion, the possibility of chaotic ray motion exists. Numerical experiments strongly support the expectations that, under such conditions, ray trajectories exhibit chaotic behavior (see, e.g., Brown et al., 1991; Smith et al., 1992; Abdullaev and Zaslavskii, 1989). These studies, however, assume spatially periodic ocean properties. This assumption allows readily available mathematical tools to be exploited. Unfortunately, a similar set of tools is not available to treat problems involving more realistic (nonperiodic) ocean structure. Chaotic behavior, which presumably persists in some form in realistic ocean environments, is characterized by exponential growth of small errors and leads unavoidably to the conclusion that, under such conditions, predictability of ray trajectories is limited to small times.

Does this imply a lack of predictability of the corresponding wavefield? Probably not. The reason is that the ray description of the wave motion is a nonlinear approximation to a linear wave equation. (For the system described by (4) and (6) the corresponding linear wave equation is the mild slope equation—see, e.g., Mei, 1983). Because nonlinearity is a necessary condition for chaos, the linear wave equation does not admit chaotic solutions. These solutions may have different properties, however, depending on whether the corresponding ray trajectories are chaotic or not. (There is a vast literature on the corresponding quantum mechanical problem—see Reichl, 1992, for an excellent recent review.) Wavefield statistics, for example, may be very different depending on whether the corresponding ray trajectories are chaotic or not.

It should also be noted that for ocean waves the linear wave equation is itself an approximation to a nonlinear wave equation. This leads to more questions. Does the nonlinear wave equation admit chaotic solutions, and, if so, is there any connection

between this chaos and chaotic behavior in the corresponding ray trajectories? The answers are currently not known.

#### FINAL REMARKS

Our discussion of the topics in the three preceding sections led to a number of questions relating to chaotic ocean dynamics. For the most part, the questions corresponding to the different topics were not the same. This is consistent with our view that chaotic ocean physics should not be treated as a unified branch of ocean physics. Rather, results from studies of low-order dynamical systems should be thoughtfully applied to selected problems in ocean physics in a manner which complements more traditional approaches to the same problem.

Not surprisingly, we have seen that the ocean is more complicated than the systems normally studied in the context of nonlinear dynamics. This discrepancy should be viewed as a challenge to both physical oceanographers and nonlinear dynamicists; both groups stand to benefit from collaborating. The example given earlier of Osborne et al.'s (1986) work motivating studies on the relationship between stochasticity and fractal behavior is an excellent example of precisely this type of symbiotic relationship.

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#### REFERENCES

- Abdullaev, S.S., and Zaslavskii, G.M., 1989: Fractals and ray dynamics in a longitudinally inhomogeneous medium, *Sov. Phys. Acoust.*, *34*, 334-336.
- Bauer, S.T., and M.G. Brown, 1992: Empirical low-order ENSO dynamics, *Geophys. Res. Lett.*, *19*, 2055-2058.
- Bjerknes, J., 1969: Atmosphere teleconnections from the equatorial Pacific, *Mon. Wea. Rev.*, *97*, 163-172.
- Broomhead, D.S., and G.P. King, 1986: Extracting qualitative dynamics from experimental data, *Physica.*, *20 D*, 217-235.
- Brown, M.G., F.D. Tappert, and S.E.R.B. Sundaram, 1991: Chaos in small amplitude surface gravity waves over slowly varying bathymetry, *J. Fluid Mech.*, *227*, 35-46.

- Brown, M.G., and K.B. Smith, 1990: Are SOFAR float trajectories chaotic? *J. Phys. Oceanog.*, 20, 139-149.
- Brown, M.G., and K. B. Smith, 1991: Ocean stirring and chaotic low-order dynamics, *Phys. Fluids A*, 3, 1186-1192.
- Enfield, D.B., 1989: El Niño, past and present, *Rev. Geophys.*, 27, 159-187.
- Lighthill, J., 1978: *Waves in Fluids*, Cambridge University Press, Cambridge, 504 pp.
- Mei, C.C., 1983: *The Applied Dynamics of Ocean Surface Waves*, Wiley-Interscience, New York, 364 pp.
- Münnich, M., M.A. Cane, and S.E. Zebiak, 1991: A study of self-excited oscillations of the tropical ocean-atmosphere system. Part II: Nonlinear cases, *J. Atmos. Sci.*, 48, 1238-1248.
- Neelin, J.D., 1990: A hybrid coupled general circulation model for El Niño studies, *J. Atmos. Sci.*, 47, 674-693.
- Osborne, A.R., A.D. Kirwan, A. Provenzale and L. Bergamasco, 1986: A search for chaotic behavior in large and mesoscale motions in the Pacific Ocean. *Physica*, 23 D, 75-83.
- Ottino, J.M., 1990: *The Kinematics of Mixing: Stretching, Chaos and Transport*, Cambridge University Press, Cambridge.
- Reichl, L.E., 1992: *The Transition to Chaos in Conservative Classical Systems: Quantum Manifestations*, Springer, New York, 551 pp.
- Schopf, P.S. and M.J. Suarez, 1988: Vacillations in a coupled ocean-atmosphere model, *J. Atmos. Sci.*, 45, 549-566.
- Smith, K.B., M.G. Brown and F.D. Tappert, 1992: Ray chaos in underwater acoustics, *J. Acoust. Soc. Am.*, 91, 1939-1949.
- Tabor, M., 1989: *Chaos and Integrability in Nonlinear Dynamics*, Wiley-Interscience, New York, 364 pp.
- Vallis, G.K., 1986: El Niño: A chaotic dynamical system? *Science*, 232, 241-245.
- Vallis, G.K., 1988: Conceptual models of El Niño and the southern oscillation, *J. Geophys. Res.*, 93, 13,979-13,991.
- Vautard, R. and M. Ghil, 1989: Singular spectrum analysis in nonlinear dynamics with applications to paleoclimatic time series, *Physica*. 35 D, 395-424.
- Wyrtki, K., 1975: El Nino--the dynamic response of the equatorial Pacific Ocean to atmospheric forcing, *J. Phys. Oceanog.*, v, 572-584.