WEAK WAVE AND VORTEX INTERACTIONS

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ABSTRACT

We report on theoretical analysis and direct numerical simulations of internal wave and vortical mode interactions in stably stratified fluids. Utilizing multiple-time-scale perturbation analysis, a resonant interaction is found between a vortical mode and two internal waves in which the vortical mode plays the role of a catalyst. This interaction could cause significant modification of the internal wave field. Vortical mode self-interactions are found to be strongly nonlinear, and can be a significant source of internal waves. The numerical simulations indicate that the theoretical analysis is valid for the small parameter (Froude number) of order one or less. Furthermore, in each case computed in this range of Froude number, the vortical mode exhibited strong instabilities, transferring energy to larger horizontal scales.

1. INTRODUCTION

The traditional view of oceanographic flows at higher frequencies (between the local Coriolis and buoyancy frequencies) and smaller scales (between tens of meters up to about tens of kilometers) is that these flows mainly consist of inertial gravity waves. Laboratory visualizations have indicated, however, that quasi-horizontal meandering motions often exist superimposed upon an internal wave field. Such flows have been observed, e. g., in the later stages of decay of a turbulent wake (Lin and Pao, 1979), in the later stages of grid turbulence (Liu, 1980), and in the long-time behavior of a short duration jet (van Heijst and Flór, 1989) when such experiments are carried out in a stably stratified fluid.

In an attempt to explain the laboratory results and also some numerical simulation data, Riley et al. (1981) have suggested that the presence of quasi-horizontal structures, which have been termed vortical modes (Müller et al.,

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1986), is due to the domination of the flow by stable density stratification. Introducing a Froude number defined by $F = u'/NL$, where u' is an rms velocity, N is the buoyancy frequency, and L is an integral scale (characterizing the larger-scale structure), they argued that the vortical modes appear when F becomes small, i. e., when the influence of stratification becomes strong. Furthermore, they offered scaling arguments to explain why the flow should consist of both vortical modes and internal waves when F becomes small, and gave a mathematical decomposition of the velocity field into wave and vortical mode components. Subsequently Lilly (1983) extended these ideas to large-scale geophysical flows (with rotation), in which case the small parameter is a Rossby number. Lilly suggested that vortical modes might explain recent atmospheric spectral data (Gage, 1979) at intermediate scales. Müller (1988; see also Müller et al., 1988) has proposed that vortical modes explain the vertical fine-structure observed in the ocean internal wave field.

At the present time, little is known about the properties of vortical modes at the internal wave scales, for example, the vortical mode energy levels, spectral distributions, dynamical interactions, sources of energy, and sinks of energy. The goal of the work presented here is to elucidate the dynamic interactions of the vortical modes, and in particular: (i) to identify the principal interactions which affect vortical modes; (ii) to determine how weakly nonlinear theory must be modified to take into account these interactions; and (iii) to test the resulting theoretical predictions by comparison with results from direct numerical simulations of the fundamental equations of motion.

There are two key points upon which this theory rests. The first is that a small parameter exists. This is an assumption implicit in any wave theory, and implies that, in some sense, the nonlinearities are sufficiently weak that the mathematical concept of a wave is useful. This small parameter can be taken to be, for example, the ratio of the wave period to an advective time. For higher frequency motions this ratio is a Froude number, as utilized by Riley et al. (1981), whereas for lower frequency motions it is a Rossby number (Lilly, 1983).

The second point upon which the theory rests is the use of Ertel's potential vorticity in the decomposition of the flow fields (Müller et al., 1986). We define the internal wave field to be that part of the flow which does not contribute to the potential vorticity, and the vortical mode field to be that part of the flow which accounts for all of the potential vorticity. A principal rationale for using potential vorticity in the decomposition is that, as a dynamic quantity conserved following the motion in the absence of molecular diffusion, it cannot be associated with wave propagation. This decomposition reduces to that obtained when the governing equations are linearized (Müller et al., 1986), and Staquet and Riley (1989) have shown that this decomposition can be usefully extended to weakly nonlinear flows. It should be noted that, with this definition, the vortical mode is the same as what is termed 'stratified turbulence' at intermediate atmospheric scales (Lilly, 1983), and the 'slow quasi-manifold' or the solution to the 'balanced equations' (McIntyre and Norton, 1991) at planetary scales. Furthermore Lilly (1983) has demonstrated that, at intermediate scales, the vortical mode satisfies the (nonlinear) geostrophic turbulence equations to lowest order.

In an effort to understand the vortical mode dynamics and their interaction with the internal wave field, we have recently examined the nonlinear interaction of simple monochromatic internal waves and vortical modes, in particular addressing wave-wave, wave-vortical mode, and vortical mode-vortical mode interactions (Lelong and Riley, 1991). In the next section, we briefly review the theory of Lelong and Riley and discuss their principal results. In the third section, results of direct numerical simulations of wave-vortical mode interactions are presented and compared with theory. In addition, simulations are presented for initial conditions consisting only of vortical modes. The behavior of the resulting flows is examined, and the results compared to the scaling analysis suggested by the theory. In the final section, the results are summarized and discussed.

2. RESULTS FROM PERTURBATION ANALYSIS

As mentioned in the previous section, vortical modes are visually observed in laboratory experiments when the Froude number based upon the energy containing range, i. e., $F = u'/NL$, is small. This Froude number can be interpreted as the ratio of two time scales: N^{-1}, the buoyancy period, and L/u', an advective time scale. In the experiments, the buoyancy period has thus become small compared to the advection time. F being small is also the usual requirement for linear or weakly nonlinear internal wave theory to be valid.

This suggests the use of multiple time scale perturbation analysis to analyze the dynamics of the internal waves and vortical modes. Lelong and Riley (1991) have carried out such an analysis, and begin by writing the velocity field $\vec{u}(\vec{x},t)$ in the form, as suggested by Lilly (1983):

$$\vec{u} = \nabla \wedge \psi \vec{i}_z + \nabla_H \phi + w \vec{i}_z \, . \qquad (1)$$

[Note that this analysis has also been extended to rotating flows (Lelong, 1989) by starting with the decomposition suggested by Müller et al. (1986), which is an extension of the above expression to include the effects of rotation.] Here the subscript H denotes the horizontal component. To lowest order the stream function ψ (and an associated density field) completely determine the vortical mode, while the vertical velocity w (or ϕ, where $w = -\int \nabla^2 \phi \, dz$) specifies the internal wave field. When these expressions are substituted into the Navier-Stokes equations subject to the Boussinesq approximation, an equation for ψ, namely the vertical vorticity equation, is obtained while an equation for w results from manipulation of the vertical and horizontal momentum equations. In addition, the equation for the buoyancy is required.

Velocities are nondimensionalized by u', lengths by L, and time by N^{-1}. (It is assumed that N is uniform in space.) All nondimensionalized dependent variables are expanded in power series of F, and terms of like powers of F are equated. Furthermore, a 'fast' time of the order of the buoyancy period and a 'slow' time of the order of the advective time are now both treated as independent variables. To lowest order (on the short time scale N^{-1}), linear theory is recovered. On this time scale, the vortical modes are steady. The equations at the next order give an $\mathcal{O}(F^2)$ correction to these solutions unless a resonance exists, in which case the form of the lowest order solutions has to be reassessed in order for the series expansion to remain valid. The elimination of secular terms from the second order equations yields the evolution of the solutions on the slow, advective time scale. The results depend on the specific problem considered, i. e., on the initial conditions. (Note that when this analysis is extended to the case with rotation, the small parameter is a Rossby number, and the theory proceeds in an analogous manner.)

Wave-Wave Interactions

For this case, the initial conditions are taken to be two linear internal waves and a linear vortical mode. The internal waves are arbitrarily oriented, while the vortical mode is assumed to be harmonic in space and is given in terms of a stream function as in Equation (1) above. The results of the analysis are an extension of resonant wave interaction theory in the presence of a vortical mode, and also the application of resonance theory to triads which are out of the vertical plane. Wave resonant triads are unaffected by the presence of the vortical mode, unless there exists a wave-vortical mode resonance (see the next subsection below). Furthermore, resonant triad analysis is readily extended to wave triads not lying in a vertical plane. The only deviation from the vertical plane case is that the gravity vector is effectively reduced by the cosine of the angle of the triad-containing plane with respect to the vertical.

The initial wave fields, being solutions to the linear wave equations, possess no potential vorticity. Since potential vorticity would be expected to be conserved for these weak wave interactions, the potential vorticity should remain zero. From the decomposition of the flow field discussed in the Introduction, no vortical mode would be expected to be generated. One key result of this theory, however, is that, for interactions out of a vertical plane, linearized potential vorticity can be produced at higher orders. This can lead to the erroneous conclusion that wave-wave interactions can be a source of vortical mode energy (e. g., Dong and Yeh, 1988). Potential vorticity is, however, exactly conserved at each order by such interactions so that no vortical mode energy is generated by such an interaction.

Another important related point is that, although resonances out of the vertical plane are as likely to occur as in-plane resonances, it is not clear that any of the present theories [e. g., McComas and Bretherton (1977)] include such interactions.

Wave-Vortical Mode Interactions

In this case the initial conditions consist of one internal wave (with wave number vector $\vec{\kappa}_1$ and frequency ω) and one vortical mode (with wave number vector $\vec{\kappa}_2$). The analysis predicts that a resonance can occur with a second internal wave (with wave number vector $\vec{\kappa}_3$) of the same frequency if the wave numbers satisfy

$$\vec{\kappa}_1 \pm \vec{\kappa}_2 \pm \vec{\kappa}_3 = 0\,. \tag{2}$$

The wave number vectors of the internal waves lie on a vertical cone whose surface is at angle $\theta = \cos^{-1}\omega$. (See Figure 1.) Note that this interaction could act to redistribute energy broadly in wave number space.

The nonlinear amplitude equations for this case can be solved analytically, and it is found that the role of the vortical mode is catalytic, i. e., it is needed for the interaction to occur, but it does not actively participate in the energy exchange. (This mechanism is reminiscent of the elastic scattering interaction discovered by Phillips, 1968, although the dynamics for the present case are significantly different.) The two wave modes exchange energy harmonically on the slow time scale, the vertical velocity being given by

$$w = \cos(\Gamma F t)\sin(\vec{\kappa}_1 \cdot \vec{x} - \omega t) - \sin(\Gamma F t)\cos(\vec{\kappa}_3 \cdot \vec{x} - \omega t)\,. \tag{3}$$

Here $2\pi/\Gamma F$ is the interaction period, where Γ is given by

$$\Gamma = \frac{B\kappa_1\kappa_3 \sin^2\theta \sin\Delta\gamma}{2}\left\{\cos^2\theta\cos\Delta\gamma + \sin^2\theta\right\}. \tag{4}$$

The parameter B is the amplitude of the vortical mode and $\Delta\gamma$ is the angle between the horizontal components of the wave number vectors of the two waves. Note that this same resonance exists in the case with rotation (Lelong, 1989).

Vortical Mode Interactions

In this final case, initial conditions consisting of two vortical modes have been considered. It is found that all interactions are resonant because the resonance condition imposed on the frequencies is identically satisfied, the frequency of any vortical mode being zero regardless of its wavenumber. As a consequence, the vortical mode equations are fully nonlinear on the vortical mode time scale (L/u'). It is found that, to lowest order, the vortical mode velocity, pressure, and density satisfy:

$$\frac{\partial}{\partial t}\vec{u}_H + \vec{u}_H \cdot \nabla \vec{u}_H = -\nabla p + R^{-1}\nabla^2 \vec{u}_H \quad (5a)$$

$$\nabla \cdot \vec{u}_H = 0 \quad (5b)$$

$$0 = -\frac{\partial p}{\partial z} - \rho \quad (5c)$$

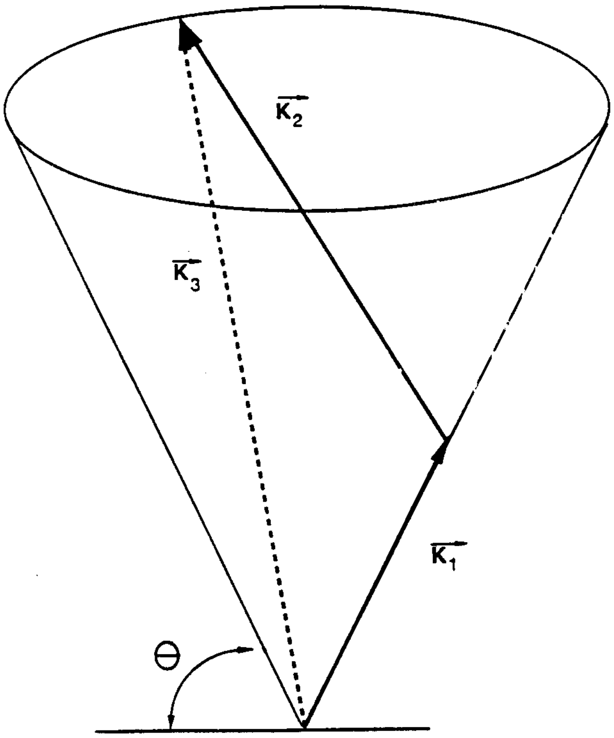

Figure 1. Wave/vortical mode resonant triad.

Here $R = u'L/\nu$ is the Reynolds number. Note that these equations describe horizontal motion in each horizontal plane, but with vertical variation retained. This result was suggested by Riley et al. (1981) using heuristic arguments. In addition to the conclusion that the vortical modes satisfy Equation (5), it was found that the vortical modes excite internal waves at higher order, with the amplitude of the internal waves scaling as F (or the energy as F^2). Note that, with rotation, the lowest order equations are equivalent to the (nonlinear) geostrophic turbulence equations (Lilly, 1983).

3. SIMULATION RESULTS

In this section we present results of direct numerical simulations and comparisons with theory for both wave-vortical mode interactions and vortical mode self-interactions. The simulations employ pseudo-spectral numerical methods with leap-frog time-stepping (smoothed every 25 time steps). The wave-vortical mode simulations were performed on $32 \times 32 \times 32$ point computational grids, while the vortical mode interactions used $64 \times 64 \times 64$ point grids. All calculations were performed on an Ardent Titan Mini-Supercomputer Server.

Wave-Vortical Mode Interactions

As discussed in the previous section, a resonance was found to exist between a vortical mode and two internal waves if the two waves are of the same frequency and the three wave number vectors form a triangle on the surface of a vertically-oriented cone. In order to examine these interactions we have performed direct numerical simulations for a number of different resonance conditions, and for a variety of different Froude numbers. The following case is typical of the results from these simulations, and represents the interaction of an internal wave with a horizontally shearing current. The latter is a special case of a vortical mode. The initial vertical velocity of the internal wave satisfies

$$w(\vec{x}, 0) = \cos(\vec{\kappa}_1 \cdot \vec{x}),$$

while the vortical mode is initially given by

$$\vec{u}(\vec{x}, 0) = \left(\kappa_{2_2} B \sin(\vec{\kappa}_2 \cdot x), -\kappa_{2_1} B \sin(\vec{\kappa}_2 \cdot x), 0\right).$$

We choose

$$\vec{\kappa}_1 = (2, 0, 2),$$

giving a wave propagating at 45° to the horizontal. The vortical mode wave number is

$$\vec{\kappa}_2 = (-2, 2, 0),$$

giving a horizontal current uniform in x_3, but which is sinusoidal in the horizontal with wave number magnitude $|\vec{\kappa}_2| = 2\sqrt{2}$. Based upon the resonance conditions [Equation (2)], the resonant wave should be found at

$$\vec{\kappa}_3 = (0, 2, 2).$$

The parameter B is taken to be 2.0, and the Froude number F to be 0.02. From the scaling analysis, this implies that the wave frequency ω in the simulations is $2\pi/F$, while the interaction frequency Γ is 2.0. The perturbation theory indicates that both the kinetic and the potential energy in the initial internal wave ($\vec{\kappa}_1$) should vary with time as $0.5\cos^2(2t)$, and that in the forced wave ($\vec{\kappa}_3$) as $0.5\sin^2(2t)$.

Figure 2 contains a plot of the potential energy at the forced wave number $\vec{\kappa}_3$ versus time taken from the simulations. Also plotted is the theoretical prediction. We see that the simulations follow the theory fairly closely for a time of about 2 to 2.5, at which point the potential energy in this wave mode begins to oscillate (near the buoyancy frequency), and the results begin to deviate significantly from the theory. More insight into this problem is given by examining the kinetic energy of each mode for this case, as shown in Figure 3. According to the theory, the kinetic energy in the vortical mode should remain uniform in time. Again we see that the simulation results follow the predicted oscillation in the wave kinetic energy up to a time of about 2 to 2.5, at which point the computed solutions rapidly diverge from the theoretical predictions. Furthermore, the kinetic energy in the vortical mode remains approximately constant up to this time, and it then begins to exhibit a large deviation from its initial value.

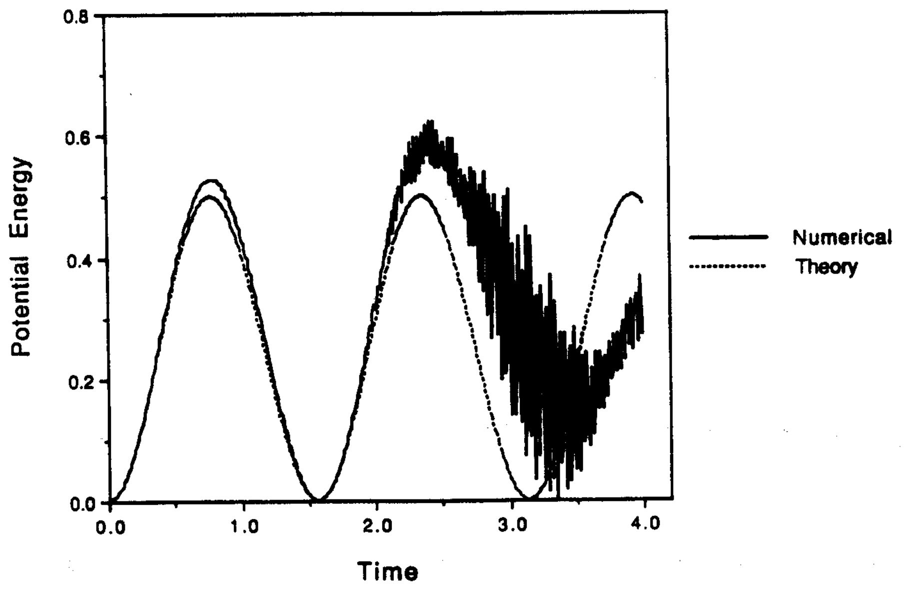

Figure 2. Potential energy in the forced wave $\vec{\kappa}_3$; $F_t = 0.02$.

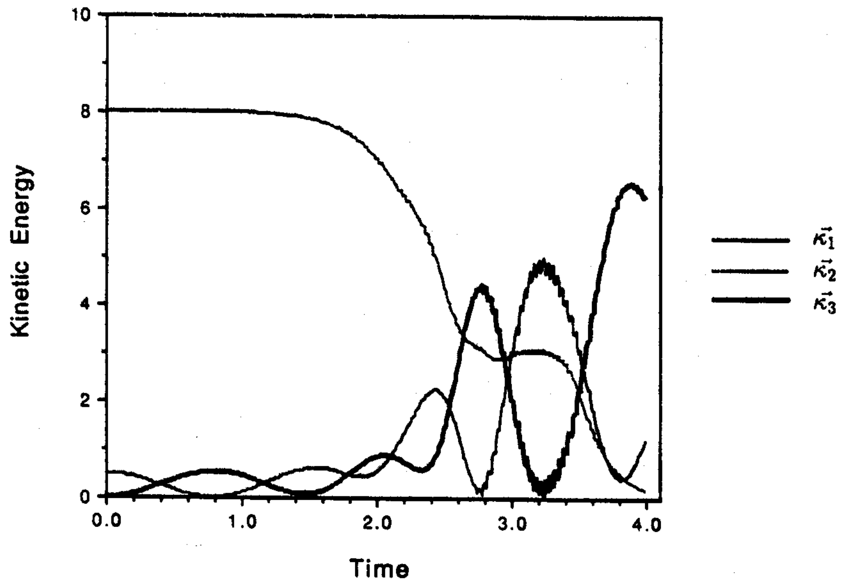

Figure 3. Kinetic energy in all three modes; $F_t = 0.02$.

Visual analysis of the results of this case determined that the horizontally shearing motion was subject to a shear instability which arose dramatically at about a time of 2. Furthermore this instability transferred energy mainly into the vortical mode components at wave numbers $\vec{\kappa}_1$ and $\vec{\kappa}_3$. Prior to this instability, however, the perturbation theory gave accurate predictions of the interactions.

Vortical Mode Interactions

In order to examine nonlinear vortical mode interactions we have considered initial conditions consisting of a sum of spatially harmonic vortical modes. The initial velocity field is given by

$$\vec{u}(\vec{x},0) = \cos x_3(\cos x_1 \sin x_2, -\sin x_1 \cos x_2, 0)\,, \tag{6}$$

while the initial perturbation density field is identically zero. For the nonstratified case this initial condition defines the Taylor-Green problem (Taylor and Green, 1937), with the velocity field oriented such that it is initially horizontal. This problem has received much attention in the literature (e. g., Orszag, 1971). The perturbation analysis predicts that as $F \rightarrow 0$, the flow field should satisfy Equation (5) to lowest order. The exact solution to these equations, satisfying the above initial conditions, is

$$\vec{u}(\vec{x},0) = e^{(-3R^{-1}t)} \cos x_3(\cos x_1 \sin x_2, -\sin x_1 \cos x_2, 0)\,. \tag{7}$$

The nonlinear terms in the momentum equation [Equation (5a)] are exactly balanced by the pressure gradient, so that the equations become linear. The streamline pattern remains independent of time as the velocity field decays due to viscosity. This solution is an extension of a well-known solution for the two-dimensional Taylor-Green problem (see, e. g., Staquet, 1985).

We performed a series of simulations for a number of different Froude numbers, using the above initial conditions for each case. For all the cases presented the initial Reynolds number R was fixed at 200. Figure 4 gives a plot of the volume integrated kinetic energy of the horizontal velocity (normalized by its initial value) for these different cases. The case with $F = \infty$ corresponds to the Taylor-Green problem, and agrees with previously published simulation results. We see that increasing the stratification decreases the decay rate, as might be expected.

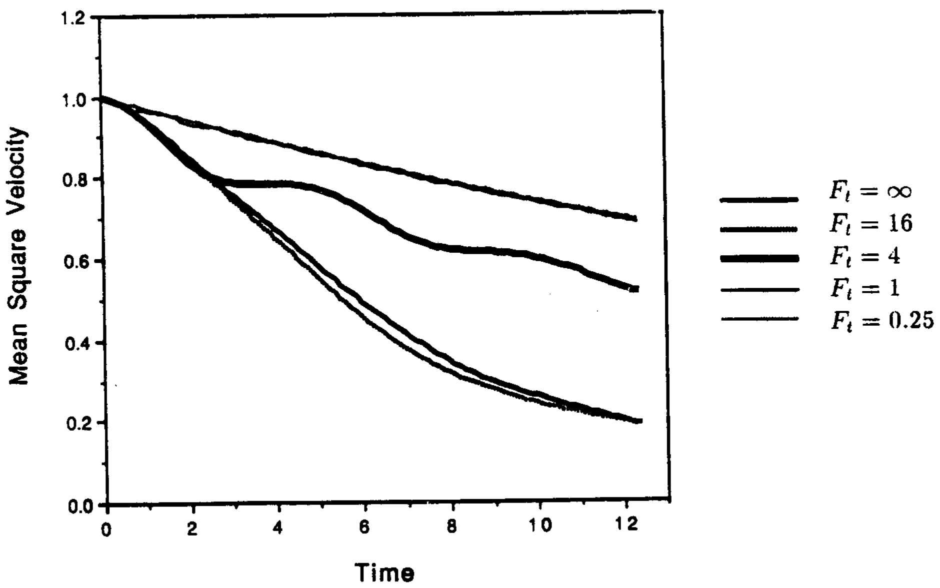

Figure 4. Taylor-Green problem: normalized kinetic energy of horizontal velocity for various Froude numbers.

Furthermore as F becomes small, roughly $F \leq 1.0$, the computed solutions coincide, as expected from the perturbation theory. Figure 5 gives a plot of the same data on an expanded scale, for the cases with stronger stratification. Also plotted is the result from the theoretical solution, Equation (7). We see that the solution to the predicted asymptotic equations agrees very well with the simulation results for $F \leq 1.0$.

Figure 6 contains plots of the wave energy for these same cases. The wave energy contains the potential energy plus the wave part of the kinetic energy, as defined by Equation (1). Note that the wave energy decreases significantly as F is

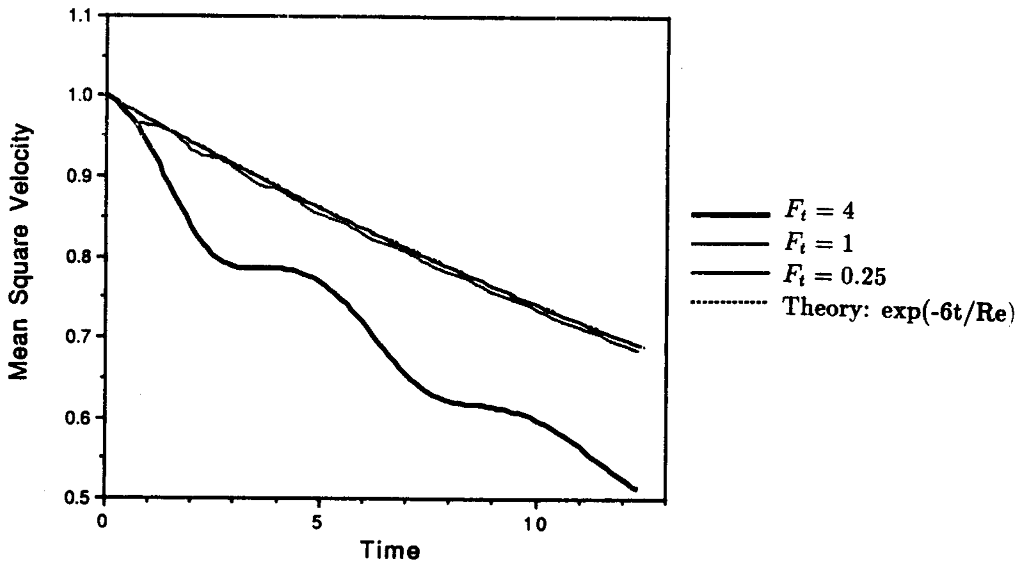

Figure 5. Taylor-Green problem: normalized kinetic energy of horizontal velocity for several Froude numbers: expanded scale.

decreased. When these results are replotted, scaled with F^{-2}, as suggested by the theory, then the wave energy collapses well for $F \leq 4.0$ (Figure 7). Therefore, for $F \leq 1.0$, the perturbation analysis is consistent with the results for both the horizontal kinetic energy, which consists mainly of the vortical mode, and the wave energy.

This present case is somewhat degenerate because, as F becomes small, the nonlinear and horizontal pressure gradient terms come into balance, leading to the simple viscous decay given by Equation (7). As mentioned, this is similar to the result for the two-dimensional Taylor-Green problem. It is well-known for the two-dimensional problem, however, that the solution is unstable to subharmonic perturbations. The length scales of the flow continually grow larger as energy is nonlinearly transferred to lower wave numbers. Therefore we also performed a series of simulations for different F using the above initial conditions, Equation (6), but with white noise added. Figure 8 gives a sequence of constant contours of ψ in a horizontal plane for the case $F = 1.0$. At this value of stratification these contours approximate streamlines in a horizontal plane. We see that the flow develops nonlinearly in time, as energy is continually transferred to larger scales, reminiscent of the two-dimensional problem. Figure 9 contains plots of ψ in two different horizontal planes at a later time. We see that the two layers have become decoupled, the flow in the two planes being very different. Finally, Figure 10 has plots of the total energy versus time for the different cases computed. Again the results converge for $F \leq 1.0$, consistent with the predictions of the perturbation theory.

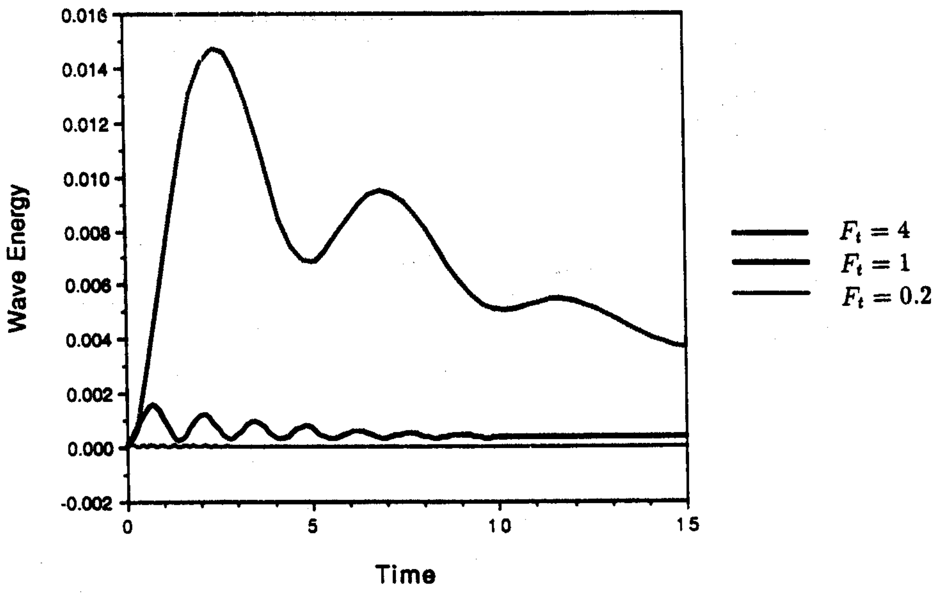

Figure 6. Taylor-Green problem with white noise: wave energy for three different Froude numbers.

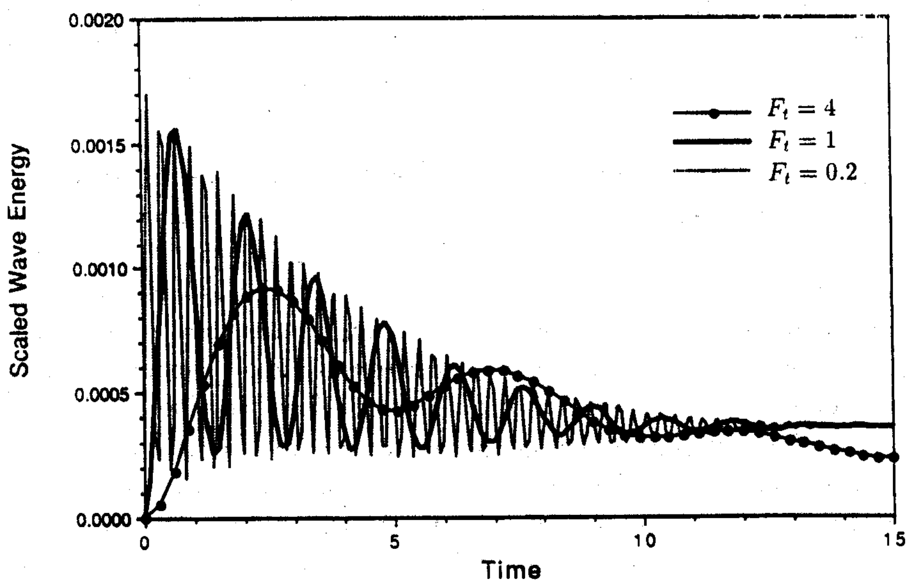

Figure 7. Wave energy scaled by $1/F_t^2$ for three different Froude numbers.

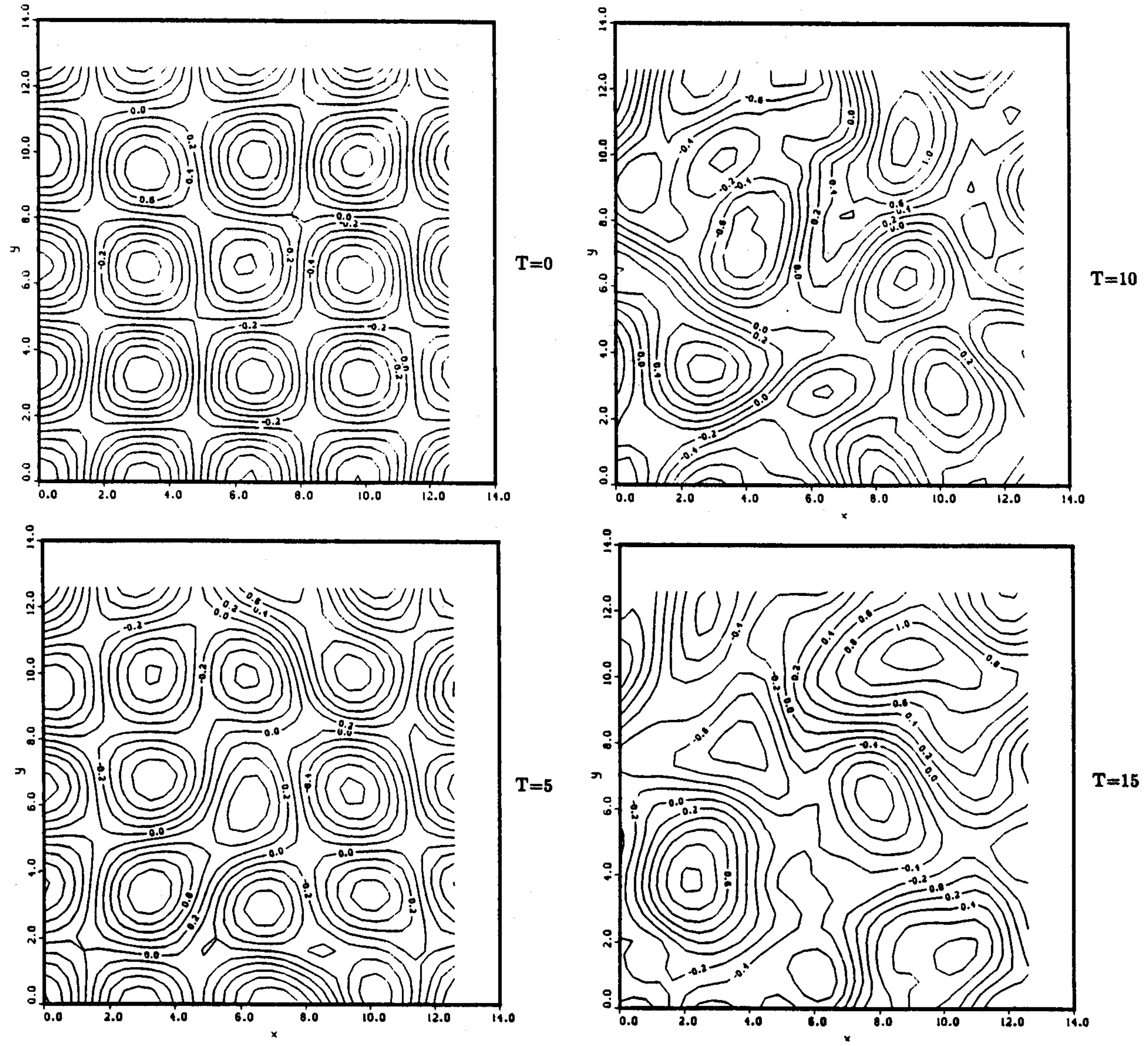

Figure 8. Sequence of constant contours of the stream function in a horizontal plane for $F_t = 0.2$ at T=0, T=5, T=10, and T=15.

4. CONCLUSIONS AND DISCUSSION

We have reported on theoretical analysis and direct numerical simulations of internal wave and vortical mode interactions in strongly stratified flows. The theoretical work utilizes multiple-scale expansions assuming the existence of a small parameter (e. g., the Froude number). Furthermore, a decomposition of the flow field into internal waves and vortical modes that is based upon Ertel's potential vorticity is employed. The objectives are: (i) to identify the principal interactions

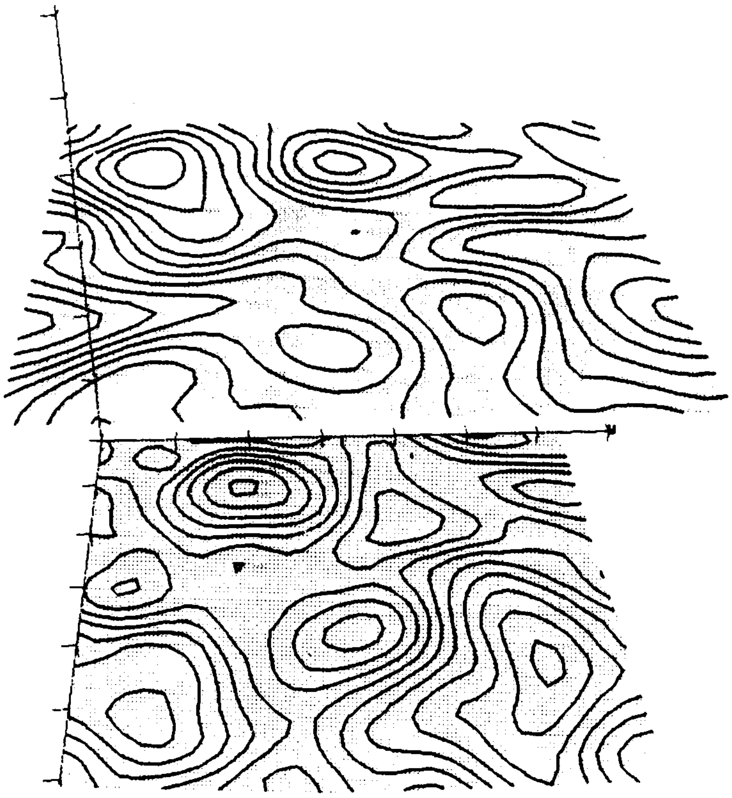

Figure 9. Constant contours of the stream function in two different horizontal planes for $F_t = 0.2$ at T=15.

which affect vortical modes ; (ii) to determine how weakly nonlinear theory must be modified to take into account these interactions; and (iii) to test the resulting theoretical predictions by comparisons with results of direct numerical simulations of the fundamental equations of motion.

Wave-wave interactions are considered theoretically, and previous results on wave resonance are reproduced. Resonances out of the vertical plane are also found, and an apparently erroneous conclusion regarding the excitation of vortical modes by internal wave interactions is explained. Another interaction examined is the resonance of two internal waves and a vortical mode. In this interaction the vortical mode plays the role of a catalyst, not exchanging energy with the waves, but being necessary for the interaction to occur. This interaction could have a significant effect on the development of an internal wave field. By comparing theoretical results with those from direct numerical simulations, we have found that the perturbation analysis predicts the interactions very well if the Froude number is small enough, approximately $F \leq 1.0$. From the numerical simulations we have also found, however, that the vortical modes considered were highly unstable, and ultimately experienced breakdown as the fluctuations in the flow increased. In all cases considered for this interaction the vortical mode consisted at least partially

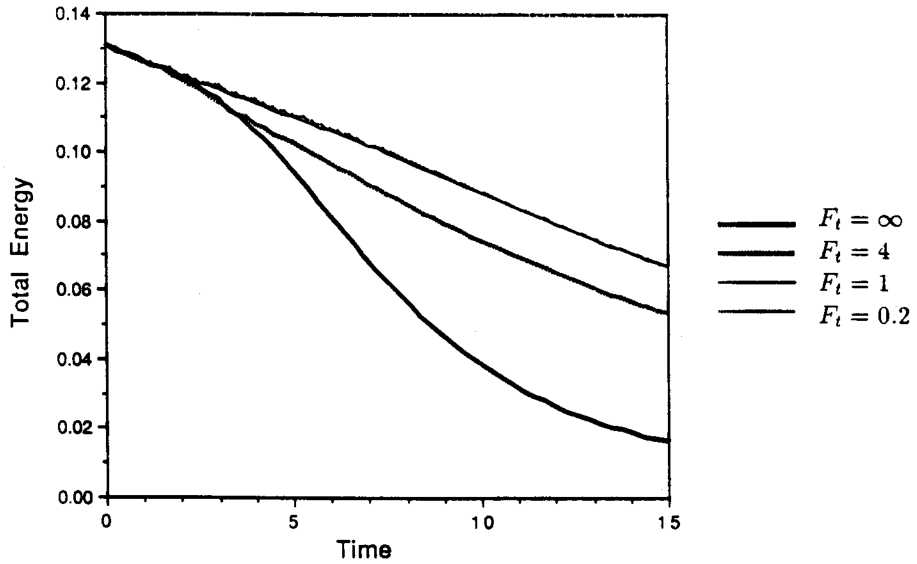

Figure 10. Taylor-Green problem: total energy for several Froude numbers.

of horizontally varying currents with multiple inflection points. These flows satisfy both Rayleigh's and Fjortoft's necessary conditions for instability (Drazin and Reid, 1981). Thus, it is not surprising that they were unstable. The instabilities appear to feed energy into the vortical modes at the same wave number as the internal waves. Once the unstable fluctuations grew to an appreciable amplitude, then the results of the simulations deviated strongly from the perturbation theory predictions.

Vortical mode self-interactions are also considered, and it is found that the lowest order governing equations are fully nonlinear. As a test case the problem of Taylor and Green is considered. An exact solution to the perturbation equations was found for this case, an extension of a well-known two-dimensional result. The results of the numerical simulations agreed well with this solution for approximately $F \leq 1.0$. Furthermore, the scaling of the potential energy predicted by the theory was also consistent with the simulation results when this condition was satisfied. When white noise was added to the initial Taylor-Green field, for small F the flow exhibited subharmonic instabilities similar to those observed for the two-dimensional Taylor-Green case. Again the perturbation equations as well as the predicted scaling of the potential energy were found to hold for approximately $F \leq 1.0$.

In both cases the simulations emphasize the importance of nonlinearity in the dynamics of the vortical modes. Furthermore, especially in the Taylor-Greene case, it is clear that upscale transfer of energy to larger horizontal scales occurs, a phenomena suggested by Lilly (1983), and observed in the laboratory (Lin and Pao, 1979) and in other numerical simulations (Riley et al., 1981; Herring and Métais, 1989; Métais and Herring, 1989). It was also clear that the different horizontal layers tend to become uncorrelated, since the horizontal flow dynamics differ in each layer. How the layers remain weakly coupled, whether through viscous effects or shear instabilities, must surely depend strongly on the Reynolds number.

If vortical modes exist on geophysical scales then their spatial characteristics and dynamical properties become of interest. Clearly, if vortical modes somewhat similar to those considered here were prevalent, then the vortical mode field would be rapidly evolving in time and probably continually be subject to shear instabilities. However, if vortical modes existed as two-dimensionally (in the horizontal) stable flows, e.g., in stable rotation satisfying Rayleigh's circulation criterion (LeLong, 1989; McWilliams, 1985), then perhaps they would be more persistent dynamically.

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