CHOOSING VARIABLES FOR INTERNAL WAVE DYNAMICS

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ABSTRACT

The issue of which variables to use for internal wave dynamics is discussed. Advantages of Hamiltonian canonical variables are mentioned. The suggestion that vertically Lagrangian coordinates be used is reviewed. It would be useful to separate internal waves form the vortical mode, but no satisfactory separation has been found.

Ten years ago, a meeting was held at Scripps Institute of Oceanography (West, 1981) at which various speakers discussed the Hamiltonian formulation of internal wave dynamics, the use of a vertically Lagrangian coordinate system, and the mode of motion now termed the vortical mode. These topics comprise the subject of this paper. In these ten years, we have not completely solved the problem; we don't really know what variables are best to use.

For the case of surface waves, the situation is much better. We have found a set of variables for which the linear approximation is remarkably good (Creamer et al, 1989). The most important short-time nonlinear features of the wave field result not from nonlinear dynamics, but simply from a nonlinear change of variables. The Hamiltonian formulation and Lagrangian coordinates played a large role in finding these variables.

Hamiltonian variables (Henyey, 1983) have a number of advantages. A very important advantage is that conservation laws are very naturally handled in that framework. If one makes an approximation, one generally does not destroy conservation laws, only changes their form slightly. That is not true of approximating equations of motion. An example of a conservation law that has been misunderstood in non-Hamiltonian frameworks is conservation of action. A commonly studied system is the dynamics of infinitesimal waves interacting with a single periodic wave, imagined to be an exact solution of the equation of motion. For this problem, in the frame of reference moving with the phase velocity of the big wave, the flow is steady. As a result, the energy, frequency, and their ratio, the action are all exactly conserved. The key is that the energy is the value of the Hamiltonian function in that frame.

Another important conservation law is momentum conservation. In non-Hamiltonian frameworks a concept of "pseudomomentum" has arisen. This concept is entirely unneeded; there is no distinction between the terms that add up to the total momentum and the "pseudomomentum". The momentian density in terms of Hamiltonian canonical variables is

$$pv = \text{linear terms} - p_1\nabla q_1 - p_2\nabla q_2 - p_3\nabla q_3 \qquad (1)$$

Each pair of variables describe one of the linear modes; p_1, q_1, are sound, p_2, q_2, are internal waves, and p_3, q_3 are vortical mode. (Only one combination of p_3 and q_3 is physical. This leads to "gauge" freedom of choosing the other, as described by Henyey, 1983.)

If for example

$$q_2 = a\cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega\tau) \qquad (2)$$

$$p_2 = b\sin(\boldsymbol{k}\cdot\boldsymbol{x} - \omega\tau) \qquad (3)$$

then the average contribution to the momentum density is

$$\langle pv\rangle = ab\mathbf{k}/2 \qquad (4)$$

The quadratic part of energy density turns out to be

$$E = ab\omega/2 \qquad (5)$$

where ω is related to $\mathbf{k}$ by the linear dispersion relation, so the action density is

$$A = ab/2 \qquad (6)$$

Another use of the Hamiltonian is in formal development. The Eikonal, or ray tracing method naturally follows from a Hamiltonian formulation (Henyey and Pomphrey, 1983). Various generalizations, such as Whitham's (Whitham, 1974) to narrow-band nonlinear waves, or the Gaussian beam method require the Hamiltonian or the closely related Lagrangian framework. Weak interaction transport theory is also most usefully derived from the Hamiltonian framework; the symmetry of the interaction coefficients which leads to detailed balance, entropy, and effective temperature is not true for arbitrary differential equations, but always occurs for Hamiltonian systems. The symmetry is obvious in a Hamiltonian derivation.

Closely related to the Hamiltonian formulation is the Lagrangian variational formulation. The canonical variational principle is that the action

$$S = \int\left(\sum P_j\dot{q}_j - H\right) \qquad (7)$$

is stationary under independent variation of all p's and q's. Unfortunately, the stationary value is always a saddle point, never a minimum. In certain special cases, the problem can be constrained (such as by assuming steady state) to make the stationary point an extremum. Arnold's stability method, and other stability calculations are examples.

Finally, the Hamiltonian can be helpful for finding new variables. The theory of canonical transformations and the Hamilton-Jacobi equation can lead to better variables. In the surface wave case we found the variables by implementing the removal of nonresonant interactions, which always can be accomplished.

Variables can be a better choice to use if the linear approximation is more accurate. There are several ways of telling if the linearization might be better. One way is that the high frequency or wavenumber part of the spectrum might be smaller due to the absence of bound harmonics; the free waves are present in either case. Another way is that the Doppler shift might be smaller, so the dispersion relation is better satisfied. A theoretical way of determining that linearization is better is by the size of the nonlinear interaction terms. By these criteria, the replacement of density as a variable by vertical displacement that is referred to as removing "fine structure contamination," or in other words, reducing bound harmonics, is an improved choice of variables.

One possible improvement in the choice of variables is the use of the vertically Lagrangian coordinate system. This idea is not new; both Ripa and Milder in West (1981) mention it. More recently Odulo (1989) has written a paper advocating its use, but I do not understand his reasons for believing it to be superior. Sherman and Pinkel (1991) have analyzed internal wave data in terms of this coordinate system. Unfortunately, the results are not completely convincing. There is not good evidence that bound harmonics were reduced. They did find that the small waves were significantly reduced in frequency (Sherman used this for the cover picture of his thesis, see Figure 1). This presumably is associated with a reduction of the Doppler shift, but of course the dispersion relation cannot be evaluated in the absence of knowing the horizontal wavenumber.

What can be said theoretically? Not much. The potential energy becomes exactly quadratic, but nonlinearities associated with stratification have always been believed unimportant relative to the advective nonlinearity. The vertical advective terms are missing, but why shouldn't horizontal advection be equally important? In order to stimulate consideration of these variables, I present the equation of motion:

The new vertical coordinate is

$$z' = z - \zeta \tag{8}$$

where ζ is the vertical displacement. The advective time derivative is

$$d_t = \partial_t' + \boldsymbol{u} \cdot \nabla' \tag{9}$$

where

$$\nabla' = \boldsymbol{e}_x \partial_x' + \boldsymbol{e}_y \partial_y' \tag{10}$$

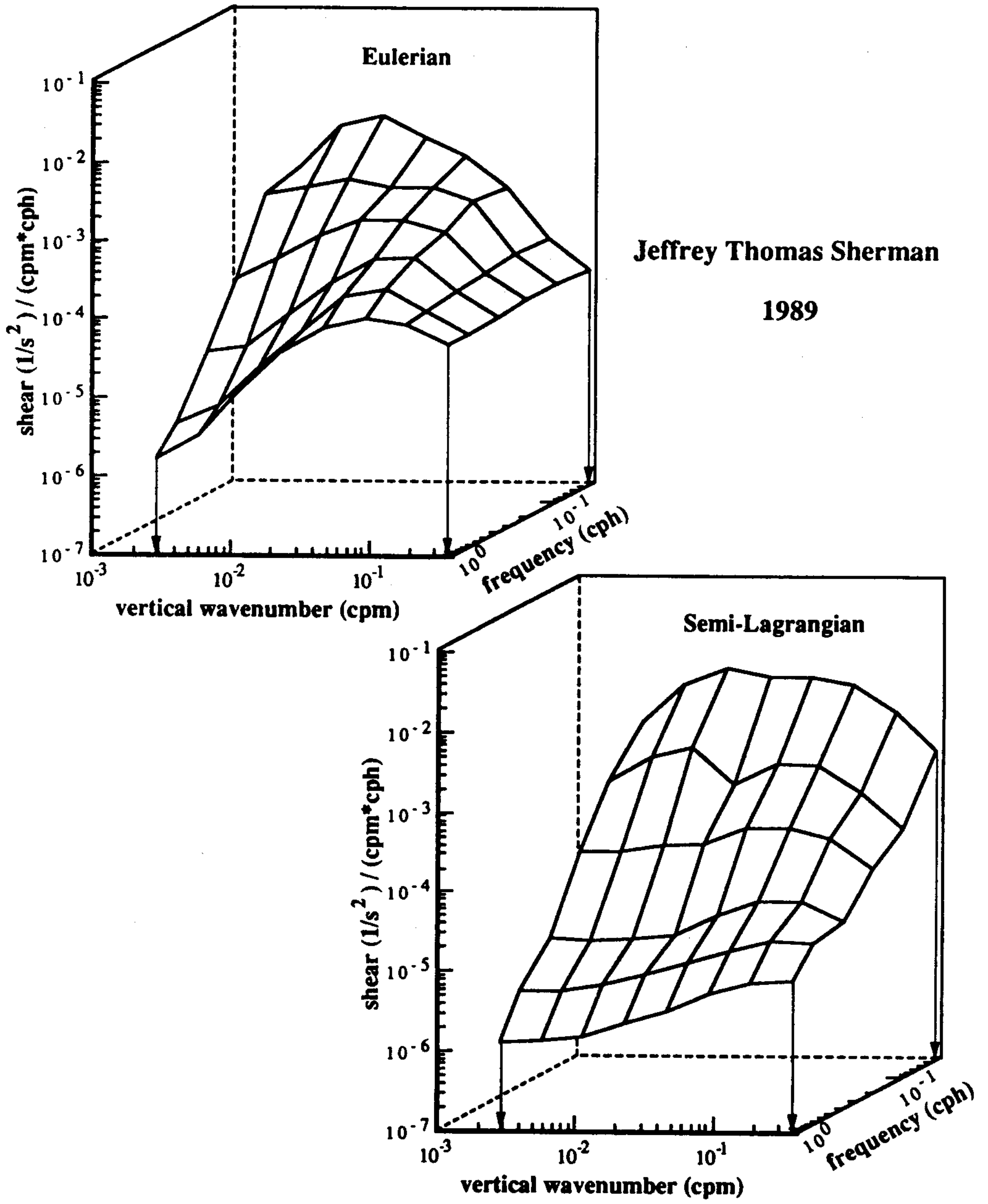

Figure 1. The cover of Sherman's Ph.D. thesis. This shows the change in the spectrum going from the Eulerian to the vertically Lagrangian coordinate system.

has only horizontal advection and **u** is the horizontal velocity. The Jacobian of the transformation is

$$S = \frac{1}{1+\partial_z{}'\zeta} \tag{11}$$

The equations of motion are

$$d_t\zeta = w \tag{12}$$

$$d_t w = -s(N^2_{z'}\zeta + \partial_z' p) \tag{13}$$

$$d_t \boldsymbol{u} = -\nabla'\zeta\, d_t w - \boldsymbol{f}\times\boldsymbol{u} - \nabla' p \tag{14}$$

$$s\partial_z' w + \nabla'\cdot\boldsymbol{u} = s\nabla'\zeta\cdot\partial_z'\boldsymbol{u} \tag{15}$$

The potential vorticity is

$$J = sf + (\hat{z} - s\nabla'\zeta)\cdot\nabla'\times\boldsymbol{v} \tag{16}$$

Another reason to choose particular variables is to separate the types of motion; sound, internal waves, and vortical motion. This separation is reasonably easily carried out in the linear approximation by diagonalizing an operator or a Hamiltonian form, and has been done numerous times. The linear separation demonstrates an advantage of Hamiltonian variables.

Using the usual variables, the linear vortical mode is geostrophic. With Hamiltonian variables there is no distinction between the geostrophic and cyclostrophic cases. This is closer to reality, as there is little difference other than size between an eddy which rotates in a week and one that rotates in an hour. Nonlinear separations, in general, cannot be accomplished. Sound is generally separated by setting $\nabla\cdot v=0$ while retaining the value of N^2 rather than the actual stratification. This is adequate for most purposes, but not ideal.

The separation of internal waves from vortical mode is much less satisfactory. A pure internal wave field can be defined by the condition that the potential vorticity

$$J = (\boldsymbol{f} + \omega)\cdot\nabla(z - \zeta) \tag{17}$$

is a constant (such as f). Even for a pure oceanic internal wave field of typical strength, the nonlinear terms in J are of comparable magnitude to the linear ones. It is clear that the linear separation of IWs from vortical mode is unsatisfactory. A step in the right direction has been taken by Staquet and Riley (1989). They define that part of the velocity belonging to the vortical mode by a prescription which is equivalent to minimizing the energy subject to specified J and ζ fields. This prescription obviously does not separate ζ. One could say that ζ is purely internal wave, but a field with $J \neq 0$ and $\zeta=0$ would radiate IWs in a time of order N^{-1}. A stationary eddy, which one would like to say is purely vortical, has nonzero ζ.

An alternative would be to say that J is one of the coordinates. It cannot be a canonical coordinate, as canonical fields have Poisson brackets that are delta function for canonical pairs, and vanish for a single field. Potential vorticity has the Poisson bracket (Abarbanel, private communication).

$$\{J(x),J(x')\} = \nabla J(x) \times \nabla \delta(x-x') \cdot \nabla(z-\zeta) \tag{18}$$

Even worse than not vanishing, this PB involves ζ which, (as discussed above) although not entirely IW, is mostly IW.

Having reviewed the choice of variables for internal wave dynamics, it is clear that the situation is not entirely satisfactory. For the surface wave problem, we are able to demonstrate variables and list their wonderful properties. For the IW problem, we find difficulties with all proposed schemes. This situation is indicative of the more general situation that internal wave dynamics is more difficult than that for other wave fields such as sound and surface waves. Progress has been made, however, and we are able at least to try out various possibilities.

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