Symmetry Preserving Mode Truncations of Inviscid Geophysical Fluid Dynamical Equations

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Abstract

We investigate the role of potential vorticity in nearly two dimensional flows of importance in geophysical fluid dynamics. Potential vorticity conservation arises from particle interchange symmetry in the Lagrangian formulation of fluid dynamics and is associated with an infinite dimensional symmetry group. In truncating the number of degrees of freedom of these fluid flows, as one does when making numerical integrations of the theory, it is not possible to keep the full infinite dimensional symmetry group. We show, in the context of the shallow water equations, how to modify the symmetry algebra and construct a Hamiltonian for the fluid which preserves the maximum symmetry consistent with the finite number of retained degrees of freedom and which becomes the original fluid as the number of degrees of freedom increases to infinity. The construction is done in planar geometry without rotation, but it also goes through for f or β plane settings, for flows on a sphere (rotating or not) and for stratified fluids. The latter application includes both internal and surface gravity waves.

1 Introduction

This is a talk about methods of truncating or restricting the number of degrees of freedom in equations of motion of relevance to geophysical fluid dynamics while preserving the symmetries leading to conservation laws respected by those evolution equations. In particular, the symmetry which will concern us here is that of particle relabeling in

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Lagrangian coordinates [Eckart 1960] which leads to conservation of potential vorticity in either Lagrangian or Eulerian formulations of the theory. The results provide a consistent mode truncation of the full continuum theory which preserves the invariance of as many of the conserved quantities of the continuum theory as is consistent with the number of retained degrees of freedom. Further, as the number of degrees of freedom goes to infinity, the original continuum theory is recovered and the full set of conserved quantities is recovered as well. This provides the possibility of reducing the number of degrees of freedom of a continuum geophysical fluid dynamics flow to a finite number, the only situation which can be treated numerically, and still preserving the maximum possible symmetry of the underlying theory.

The methods we present here are Hamiltonian, and the fluid dynamics is cast in Lagrangian realization. The advantage of this is that the underlying Lagrangian theory is canonical in the classical mechanics sense and the symmetries of the theory are manifest and easy to deal with. The corresponding Eulerian theory is non-canonical and the symmetries are hidden or "mysterious". The reason for this disguise of the symmetries is that the Eulerian theory is "reduced" from the Lagrangian formulation by considering the flow only on hypersurfaces in phase space where the conserved quantities are constant. The manifestation of these symmetries in terms of conserved quantities seems unnatural in Eulerian formulation while appearing quite natural in Lagrangian formulation.

An outline of this talk is as follows:

- (1) Lagrangian formulation of the Shallow Water Equations and the Internal Wave Equations
- (2) Invariance under particle interchange and potential vorticity conservation.
- (3) Truncating the Fourier modes and SU(N) symmetry
  - Algebra of Symmetry Generators and Dynamical Variables
  - Conserved Quantities
- SU(N) symmetric Hamiltonian, H_N; $N \to \infty$ leads to usual equations

In the SU(N) symmetric truncated theory there are $\approx N$ conserved quantities. As $N \to \infty$, we recover the continuum theory and an infinite number of conserved quantities.

This talk focuses on the shallow water equations [Pedlosky 1979] because it is for these that we have concrete results at this time. The shallow water equations are also formulated on an f-plane, that is Cartesian or flat geometry, in this talk. We know how to extend the results to flows on the surface of a sphere, but the algebra is difficult and will be reported elsewhere. In progress is work on extending these results to quasi-two dimensional geophysical flows including internal waves on a plane (f or β) and surface gravity waves. The reader will see that our methods are generally applicable to flows with a conserved potential vorticity. If there is driving and damping also present in the physical setting, then we can regard the work here as establishing a finite set of coordinates for such dynamics. When the driving or damping is not significant, then in the coordinates we present the required conservation laws are respected automatically. In that sense they provide a rational choice of truncated modes for all numerical work on geophysical problems where quasi-two dimensionality is a feature.

Our motivation for concentrating on potential vorticity modes is two fold:

- The work of Müller and co-workers [Müller 1988a, Müller 1988b] has provided evidence for the geophysical importance of potential vorticity carrying motions even at small scales.
- Conserved quantities are always important for constraining the allowed physical motions of a system and for checking numerical integrations of those equations of motion.

2 Lagrangian Fluids

In the description of fluids by the Lagrangian method [Abarbanel 1987] we are required to give the position of a fluid particle $\mathbf{y}(\mathbf{r},t)$ and its canonical momentum $\mathbf{p}(\mathbf{r},t)$ for each particle label $\mathbf{r}$, which is a two or three dimensional continuum of labels for particles. The evolution equations of these variables follows from an Action Principle which is really just Hamilton's Principle. This states that the action S in d-dimensions:

$$S(\mathbf{y},\mathbf{p}) = \int_{t_1}^{t_2} dt \int d^d r\, \rho_0(\mathbf{r})\{\frac{1}{2}\frac{\partial \mathbf{y}(\mathbf{r},t)}{\partial t}\cdot\frac{\partial \mathbf{y}(\mathbf{r},t)}{\partial t} - \epsilon(\rho,s)\} \tag{1}$$

is stationary under changes of $\mathbf{y}(\mathbf{r},t)$ near the orbit of the system. Here the internal energy density $\epsilon(\rho,s)$ is a thermodynamic quantity from which the pressure is derived. It is a function of the density and the specific entropy. In this expression for the action all partial derivatives with respect to time are with $\mathbf{r}$ held fixed. The density $\rho(\mathbf{y})$ is

$$\rho(\mathbf{y}(\mathbf{r},t)) = \rho_0(\mathbf{r})\frac{\partial(\mathbf{r})}{\partial(\mathbf{y}(\mathbf{r},t))}, \tag{2}$$

and $s = s(\mathbf{r})$ is the entropy per unit volume. Varying S with respect to $\mathbf{y}(\mathbf{r},t)$ with $\mathbf{r}$ and t fixed leads to the equations of motion

$$\begin{aligned} \rho(\mathbf{y})\frac{\partial^2\mathbf{y}(\mathbf{r},t)}{\partial t^2} &= -\nabla p(\mathbf{y},t), \\ \frac{\partial\rho(\mathbf{y},t)}{\partial t} &= -\rho(\mathbf{y})\nabla_{\mathbf{y}}\cdot(\frac{\partial\mathbf{y}(\mathbf{r},t)}{\partial t}), \\ \frac{\partial s}{\partial t} &= 0, \\ p &= \rho^2\epsilon_\rho. \end{aligned} \tag{3}$$

To reach the Eulerian formulation of fluid dynamics we identify a fixed point in space $\mathbf{x}$ with the location $\mathbf{y}(\mathbf{r},t)$ of a particular fluid particle at time t. This defines a particular label $\mathbf{R}(\mathbf{x},t)$ which identifies the fluid particle which arrives at $\mathbf{x}$ at the appointed time, so

$$\mathbf{x} = \mathbf{y}(\mathbf{R}(\mathbf{x},t),t), \tag{4}$$

and the Eulerian velocity $\mathbf{u}_E(\mathbf{x},t)$ is defined as

$$\mathbf{u}_E(\mathbf{x},t) = \frac{\partial\mathbf{y}(\mathbf{r},t)}{\partial t}|_{\mathbf{r}=\mathbf{R}(\mathbf{x},t)}. \tag{5}$$

The Lagrangian derivative at fixed label $\mathbf{r}$ becomes

$$\frac{\partial}{\partial t}|_{\mathbf{r}} = \frac{\partial}{\partial t}|_{\mathbf{x}} + \mathbf{u}_E(\mathbf{x},t)\cdot\nabla_{\mathbf{x}}. \tag{6}$$

The Eulerian formulation at fixed spatial points $\mathbf{x}$ is a **reduced** description of the fluid theory [Marsden 1984] since it describes flows restricted to surfaces in the fluid state space which have constant values of the conserved potential vorticity. Lagrangian fluid dynamics describes the evolution of *six* fields: the canonical coordinates $\mathbf{y}(\mathbf{r},t)$ and their canonical momenta $\mathbf{p}(\mathbf{r},t)$. Eulerian fluid dynamics describes the evolution of *five* fields: $\mathbf{u}_E(\mathbf{x},t), \rho(\mathbf{x},t)$, and the specific entropy $s(\mathbf{x},t)$. This reduction in number can be traced to the restriction of the flows to motion on constant potential vorticity surfaces, and that brings us to potential vorticity and its interpretation.

In this talk we consider the shallow water equations as our paradigm for a nearly two dimensional fluid with a conserved quantity. We wish to truncate to a finite number the continuum degrees of freedom of the fluid and to do so in a fashion which preserves a subset of the symmetry leading to potential vorticity conservation The truncated theory must become the correct continuum theory as the number of modes goes to infinity. The shallow water equations result from stationarity of the action

$$S = \int_{t_1}^{t_2} dt \int d^2r\, h_0(\mathbf{r})\{\frac{1}{2}\frac{\partial \mathbf{y}(\mathbf{r},t)}{\partial t} \cdot \frac{\partial \mathbf{y}(\mathbf{r},t)}{\partial t} - \frac{g}{2}\frac{1}{J}\}, \tag{7}$$

with $J = \frac{\partial(\mathbf{y}(\mathbf{r},t))}{\partial(\mathbf{r})}$. One can absorb the initial "height" $h_0(\mathbf{r})$ into the definition of the labels $\int d^2r h_0(\mathbf{r}) \to \int d^2r$ without any loss of generality, and we do that to simplify our formulae. The canonical momentum is defined in the usual way as the derivative of the Lagrangian $S = \int_{t_1}^{t_2} dt \int d^2r L[\mathbf{y}(\mathbf{r},t), \partial_t \mathbf{y}(\mathbf{r},t)]$ with respect to $\partial_t \mathbf{y}(\mathbf{r},t)$, so $\mathbf{p}(\mathbf{r},t) = \partial_t \mathbf{y}(\mathbf{r},t)$. The shallow water Hamiltonian is then

$$H(\mathbf{y},\mathbf{p}) = \frac{1}{2}\int d^2r\,[|\mathbf{p}(\mathbf{r},t)|^2 + \frac{g}{J}], \tag{8}$$

and the equations of motion follow from the Poisson bracket relation

$$\frac{\partial \bullet}{\partial t} = \{\bullet, H(\mathbf{y},\mathbf{p})\}, \tag{9}$$

using the fundamental Poisson bracket

$$\{y_a(\mathbf{r},t), p_b(\mathbf{r}',t)\} = \delta_{ab}\delta^2(\mathbf{r}-\mathbf{r}'). \tag{10}$$

Under particle interchanges which preserve the density (or $h_0(\mathbf{r})$ here) the action is invariant. This is formally expressed by requiring that $\delta_{\mathbf{r}} S = 0$ with $\mathbf{y}(\mathbf{r},t)$ and density held fixed, and was pointed out first by Eckart in 1960 [Eckart 1960]. The conserved quantity which results from this symmetry of the action is the potential vorticity

$$q(\mathbf{r},t) = \epsilon_{ab}\frac{\partial \mathbf{y}_\alpha(\mathbf{r},t)}{\partial r_a}\frac{\partial \mathbf{p}_\alpha(\mathbf{r},t)}{\partial r_b}, \tag{11}$$

and

$$\frac{\partial q(\mathbf{r},t)}{\partial t} = 0. \tag{12}$$

Translating this into Eulerian variables using the prescription given above results in

$$q_E(\mathbf{x},t) = \frac{\hat{z}\cdot\nabla_{\mathbf{X}}\times\mathbf{u}_E(\mathbf{x},t)}{h_E(\mathbf{x},t)}, \tag{13}$$

and

$$(\frac{\partial}{\partial t} + \mathbf{u}_E\cdot\nabla_{\mathbf{X}})q_E(\mathbf{x},t) = 0. \tag{14}$$

$h_E(\mathbf{x},t)$ is the usual Eulerian fluid thickness in shallow water theory and comes from the Lagrangian quantity $h(\mathbf{r},t) = \frac{1}{J}$.

These conservation laws lead to the statements that for arbitrary functions G

$$\frac{\partial}{\partial t}\int d^2r\, G(q(\mathbf{r},t)) = 0, \tag{15}$$

and

$$\frac{\partial}{\partial t}\int d^2x\, h_E(\mathbf{x},t)G(q_E(\mathbf{x},t)) = 0. \tag{16}$$

These are an infinite number of conserved quantities associated with the local particle interchange symmetry. Next we examine the algebra associated with this symmetry noting it is $q(\mathbf{r},t)$ which is the infinitesimal generator of the symmetry.

Before delving into the algebra let us make the connection with internal wave dynamics. For internal waves the flows are three dimensional and the Hamiltonian is

$$H(\mathbf{y},\mathbf{p}) = \int d^3r\,\{\frac{|\mathbf{p}-\rho_0\hat{\mathbf{R}}|^2}{2\rho_0} + \rho_0 g y_3(\mathbf{r},t) + \rho_0\epsilon(\rho)\}, \tag{17}$$

where $\hat{\mathbf{R}}$ is the rotational potential whose curl is $\hat{z}f(\mathbf{r})$, and the initial density $\rho_0(r_3)$ is taken to depend on the vertical coordinate only. The quantity conserved under particle interchange for this theory is

$$q(\mathbf{r},t) = \sum_{\alpha=1}^{3}\{\frac{\partial y_\alpha(\mathbf{r},t)}{\partial r_1}\frac{\partial}{\partial r_2}[\frac{\partial y_\alpha(\mathbf{r},t)}{\partial t} + \hat{\mathbf{R}}(\mathbf{y})_\alpha] - (r_1 \leftrightarrow r_2)\}, \tag{18}$$

or in Eulerian variables

$$q_E(\mathbf{x},t) = (\hat{z}f + \nabla\times\mathbf{u}_E)\cdot\nabla\rho. \tag{19}$$

3 Algebra of Particle Interchange Symmetry

To exhibit the algebra associated with the particle interchange symmetry of the shallow water equations, it is easier to go from configuration space $\mathbf{r}$ to Fourier space [Abarbanel 1991]. For this we place the configuration space in a box of size $L \times L$ and define Fourier transforms via

$$\begin{aligned} f(\mathbf{r}) &= \sum_{\mathbf{n}=-\infty}^{+\infty} F(\mathbf{n}) \exp[i\kappa \mathbf{n} \cdot \mathbf{r}], \\ F(\mathbf{n}) &= \frac{1}{L^2} \int d^2 r f(\mathbf{r}) \exp[-i\kappa \mathbf{n} \cdot \mathbf{r}]. \end{aligned} \tag{20}$$

Here the vector $\mathbf{n} = [n_1, n_2]$ with n_i are integers $n_i = 0, \pm 1, \pm 2, , \ldots, \pm\infty$, and $\kappa = \frac{2\pi}{L}$. With this Fourier transform pair we define $\mathbf{Y}(\mathbf{n}, t)$ and $\mathbf{P}(\mathbf{n}, t)$ as

$$\begin{aligned} y_\alpha(\mathbf{r}, t) &= \frac{1}{L} \sum_{\mathbf{n}=-\infty}^{+\infty} Y_\alpha(\mathbf{n}, t) \exp[i\kappa \mathbf{n} \cdot \mathbf{r}], \\ p_\alpha(\mathbf{r}, t) &= \frac{1}{L} \sum_{\mathbf{n}=-\infty}^{+\infty} P_\alpha(\mathbf{n}, t) \exp[i\kappa \mathbf{n} \cdot \mathbf{r}], \end{aligned} \tag{21}$$

with the normalization chosen so the fundamental Poisson bracket becomes

$$\{Y_\alpha(\mathbf{n}, t), P_\beta(\mathbf{m}, t)\} = \delta_{\alpha\beta} \delta_{0,\mathbf{m}+\mathbf{n}}. \tag{22}$$

The Fourier components of the potential vorticity are taken as

$$q(\mathbf{r}, t) = \frac{(2\pi)^2}{L^4} \sum_{\mathbf{n}} Q(\mathbf{n}, t) \exp[i\kappa \mathbf{n} \cdot \mathbf{r}], \tag{23}$$

which leads to

$$Q(\mathbf{n}, t) = \sum_{\mathbf{m},\mathbf{m}'} \mathbf{m}' \times \mathbf{m} \, P_\alpha(\mathbf{m}) Q_\alpha(\mathbf{m}') \delta_{\mathbf{n},\mathbf{m}+\mathbf{m}'}, \tag{24}$$

where $\mathbf{m}' \times \mathbf{m} = m'_1 m_2 - m'_2 m_1$ is the z component of the cross product among vectors. With these definitions of Fourier components we can easily evaluate the Poisson brackets of the $Q(\mathbf{n})$ which are the generators of the local particle interchange symmetry with the $\mathbf{Y}(\mathbf{n})$, the $\mathbf{P}(\mathbf{n})$, and themselves. This leads to

$$\begin{aligned} \{Q(\mathbf{n}), Y_\alpha(\mathbf{m})\} &= \mathbf{n} \times \mathbf{m} \, Y_\alpha(\mathbf{m} + \mathbf{n}), \\ \{Q(\mathbf{n}), P_\alpha(\mathbf{m})\} &= \mathbf{n} \times \mathbf{m} \, P_\alpha(\mathbf{m} + \mathbf{n}), \\ \{Q(\mathbf{n}), Q(\mathbf{m})\} &= \mathbf{n} \times \mathbf{m} \, Q(\mathbf{m} + \mathbf{n}), \end{aligned} \tag{25}$$

so the algebra of the $Q(\mathbf{n})$ closes, as it must if we have a symmetry, and the Fourier components of the $\mathbf{y}$ and the $\mathbf{p}$ transform under the algebra as "vectors". The factors of

$\mathbf{m} \times \mathbf{n}$ are the *structure constants* of the group of particle interchange symmetry.

Perhaps a more familiar example of this kind of algebra is that of three dimensional angular momentum in classical mechanics. The angular momentum $\mathbf{L} = \mathbf{q} \times \mathbf{p}$ or $L_a = \epsilon_{abc} q_b p_c$; $a, b = 1, 2, 3$ has the following Poisson brackets with the coordinates $\mathbf{q}$, the momenta $\mathbf{p}$, and $\mathbf{L}$ which follow from the fundamental bracket $\{q_a, p_b\} = \delta_{ab}$:

$$\begin{aligned} \{L_a, q_b\} &= \epsilon_{abc} q_c, \\ \{L_a, p_b\} &= \epsilon_{abc} p_c, \\ \{L_a, L_b\} &= \epsilon_{abc} L_c. \end{aligned} \tag{26}$$

Any quantity which satisfies $\{L_a, v_b\} = \epsilon_{abc} v_c$ is a vector under three dimensional rotations which are generated by $\mathbf{L}$. The dot product $\mathbf{v} \cdot \mathbf{v}$ is unchanged under rotations since $\{L_a, \mathbf{v} \cdot \mathbf{v}\} = 0$, and $\mathbf{L} \cdot \mathbf{L}$ is the invariant of the algebra of the rotation group. Rotational invariance of the dynamics of a system is guaranteed by having an Hamiltonian $H(\mathbf{p}, \mathbf{q})$ which satisfies

$$\{L_a, H(\mathbf{p}, \mathbf{q})\} = 0. \tag{27}$$

This also leads to the conservation (under evolution in time under $H(\mathbf{p}, \mathbf{q})$) of $L^2 = \mathbf{L} \cdot \mathbf{L}$ and any function of L^2.

A critical aspect of the angular momentum algebra which we must establish for our particle interchange algebra is that the Poisson brackets satisfy the Jacobi identity

$$\{L_a, \{L_b, L_c\}\} + \{L_b, \{L_c, L_a\}\} + \{L_c, \{L_a, L_b\}\} = 0, \tag{28}$$

for this guarantees that a combination of rotations is also a rotation and that under evolution through a finite time under $H(\mathbf{p}, \mathbf{q})$ angular momentum is conserved.

Now we return to the shallow water equations. The final ingredient we require for constructing the truncated Hamiltonian for shallow water flow is the Fourier decomposition of the Jacobian and the transformation properties of these Fourier coefficients under particle interchange. This decomposition is easily established to be

$$\begin{aligned} J &= \frac{\partial(\mathbf{y}(\mathbf{r}, t))}{\partial(\mathbf{r})} \\ &= \frac{(2\pi)^2}{L^4} \sum_{\mathbf{n}} \rho(\mathbf{n}, t) \exp[i\kappa \mathbf{n} \cdot \mathbf{r}], \\ \rho(\mathbf{n}) &= \frac{1}{2} \sum_{\mathbf{m}, \mathbf{m}'} (\mathbf{m}' \times \mathbf{m}) \mathbf{Y}(\mathbf{m}) \times \mathbf{Y}(\mathbf{m}') \delta_{\mathbf{n}, \mathbf{m}+\mathbf{m}'}, \end{aligned} \tag{29}$$

from which

$$\{Q(\mathbf{n}), \rho(\mathbf{m})\} = \mathbf{n} \times \mathbf{m}\, \rho(\mathbf{m} + \mathbf{n}) \tag{30}$$

follows. So the Fourier components of the Jacobian are also vectors under the transformations generated by the Fourier components of potential vorticity.

Just as with the three dimensional angular momentum example above, the Jacobi identity among the $Q(\mathbf{n})$ is critical in guaranteeing that finite particle interchange transformations such as are generated by finite time evolution under the shallow water Hamiltonian lead to potential vorticity conservation.

Before displaying our truncated shallow water theory we recall how the potential vorticity $Q(\mathbf{n}, t)$ is conserved in the case with an infinite number of Fourier components. For this we need to compute the Poisson bracket of $Q(\mathbf{n}, t)$ with the Hamiltonian

$$H = \frac{1}{2} \sum_{\mathbf{n}=-\infty}^{+\infty} P_\alpha(\mathbf{n}) P_\alpha(-\mathbf{n}) + \frac{g}{2} \int d^2r \frac{1}{J}. \tag{31}$$

The Poisson bracket with the first term in H is up to a factor of 2

$$\sum_{\mathbf{m},\mathbf{m}'} (\mathbf{m}' \times \mathbf{m})\, \mathbf{P}(\mathbf{m}) \cdot \mathbf{P}(\mathbf{m}') \delta_{\mathbf{n},\mathbf{m}+\mathbf{m}'}, \tag{32}$$

which vanishes because of symmetry in the $\mathbf{m}, \mathbf{m}'$ sum. The Poisson bracket with the second term is (up to a constant factor)

$$\frac{1}{\kappa^2} \int d^2r \frac{\partial(\exp[-i\kappa \mathbf{n} \cdot \mathbf{r}], J^{-1})}{\partial(\mathbf{r})}, \tag{33}$$

which vanishes by integration by parts. In a mode truncated theory the first part of this will remain: the kinetic energy will still Poisson commute with potential vorticity, but integration by parts will be absent since we will no longer have a continuum theory in label space.

4 Truncating the Number of Modes; a New Potential Vorticity Algebra

Now we restrict the number of Fourier modes allowed to the variables $\mathbf{Y}(\mathbf{n}, t)$ and $\mathbf{P}(\mathbf{n}, t)$ by keeping the Fourier sums in the bounds $-M \leq n_i \leq M$ for $i = 1, 2$. We now have N^2 degrees of freedom where $N = 2M + 1$. The fundamental Poisson bracket among the $\mathbf{Y}(\mathbf{n})$

and the $\mathbf{P}(\mathbf{n})$ is unchanged except the rule is to keep all Fourier indices within $[-M, M]$, so when $\mathbf{n}+\mathbf{m}$ appears it is to be so restricted. The problem comes when we go to the Poisson brackets of the potential vorticity $Q(\mathbf{n})$ with the coordinates or the canonical momenta or the Fourier components of the Jacobian or with itself. In this we encounter the cross product $\mathbf{m} \times \mathbf{n}$ which is the structure constant for the group action of $Q(\mathbf{n})$ in the Euclidian space of Fourier indices. By our truncation of modes we have changed the space of Fourier modes from the plane to that of a two dimensional torus; this is because we have introduced an effective periodicity in Fourier labels. To match this and preserve the Jacobi identities we replace $\mathbf{m} \times \mathbf{n}$ by

$$\mathbf{n} \times \mathbf{m} \longrightarrow \frac{1}{\kappa_N} \sin[\kappa_N(\mathbf{n} \times \mathbf{m})], \tag{34}$$

where $\kappa_N = \frac{2\pi}{N}$. Clearly as $N \to \infty$ this reduces back to the Euclidian space version $\mathbf{n} \times \mathbf{m}$. For finite N, which is our concern here, we have an effective periodicity in Fourier space now respected by the new structure constants. What is truly remarkable, however, is that this simple replacement of $\mathbf{m} \times \mathbf{n}$ also respects the Jacobi identity so a group structure is retained [Hoppe 1989].

With these new structure constants we can write the Fourier decomposition of the potential vorticity

$$Q_N(\mathbf{n}) = \frac{1}{\kappa_N} \sum_{\mathbf{m}=-M}^{M} \sin[\kappa_N(\mathbf{n} \times \mathbf{m})]\mathbf{P}(\mathbf{m}) \cdot \mathbf{Y}(\mathbf{n}-\mathbf{m}), \tag{35}$$

and for the Jacobian Fourier components, we write

$$\rho_N(\mathbf{n}) = \frac{1}{\kappa_N} \sum_{\mathbf{m}=-M}^{M} \sin[\kappa_N(\mathbf{n} \times \mathbf{m})] Y_1(\mathbf{m}) Y_2(\mathbf{n}-\mathbf{m}). \tag{36}$$

The Poisson brackets of this new $Q_N(\mathbf{n})$ with any of $\mathbf{Y}(\mathbf{n}), \mathbf{P}(\mathbf{n}), \rho_N(\mathbf{n})$ or $Q(\mathbf{n})$ takes the form

$$\{Q_N(\mathbf{n}), f(\mathbf{m})\} = \frac{\sin[\kappa_N(\mathbf{m} \times \mathbf{n})]}{\kappa_N} f(\mathbf{m}+\mathbf{n}), \tag{37}$$

with $f(\mathbf{m})$ any component of the canonical coordinates or canonical momentum or $\rho_N(\mathbf{m})$ or $Q_N(\mathbf{m})$. This set of Poisson brackets now defines a finite algebra of particle interchange transformations generated by the $Q_N(\mathbf{n})$. It also defines anything which transform as $f(\mathbf{m})$ here as a vector under this new transformation group. The group structure is guaranteed by satisfying the Jacobi identity, the demonstration of which is a tedious task left to the dedicated reader. Our job now is to establish a Hamiltonian $H_N(\mathbf{Y},\mathbf{P})$ in these truncated variables which Poisson commutes with $Q_N(\mathbf{n})$ and becomes just the shallow water Hamiltonian as $N \to \infty$. The easiest method is to seek **invariants** of the finite particle interchange algebra (it happens to be SU(N)) and construct H_N out of them.

4.1 Invariants of the Truncated Algebra

The idea is to use the transformation properties of **vectors** $f(\mathbf{m})$ under the $Q_N(\mathbf{n})$ algebra

$$\{Q_N(\mathbf{n}), f(\mathbf{m})\} = \frac{\sin[\kappa_N(\mathbf{m}\times\mathbf{n})]}{\kappa_N} f(\mathbf{m}+\mathbf{n}), \tag{38}$$

to form "dot products" $C_p(f)$ such that

$$\{Q_N(\mathbf{n}), C_p(f)\} = 0. \tag{39}$$

The $C_p(f)$ made out of powers of $f(\mathbf{m})$ are

$$\begin{aligned} C_2(f) &= \sum_{\mathbf{m},\mathbf{m}'=-M}^{M} f(\mathbf{m})f(\mathbf{m}')\delta_{0,\mathbf{m}+\mathbf{m}'} \\ C_3(f) &= \sum_{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3} f(\mathbf{n}_1)f(\mathbf{n}_2)f(\mathbf{n}_3)\delta_{0,\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3}\exp[i\kappa_N(\mathbf{n}_1\times\mathbf{n}_2+\mathbf{n}_1\times\mathbf{n}_3+\mathbf{n}_2\times\mathbf{n}_3)] \\ &\vdots \\ C_{L+1}(f) &= \sum_{\mathbf{n}_1\ldots\mathbf{n}_L}\prod_{\alpha<\beta}\exp[i\kappa_N(\mathbf{n}_\alpha\times\mathbf{n}_\beta]\, f(\mathbf{n}_1)f(\mathbf{n}_2)\ldots f(\mathbf{n}_L)f(-(\mathbf{n}_1+\mathbf{n}_2+\ldots\mathbf{n}_L)). \end{aligned} \tag{40}$$

So these are generalized "powers of vectors".

The kinetic energy term in the truncated Hamiltonian

$$KE_N = \frac{1}{2}\sum_{\mathbf{n}=-M}^{M} \mathbf{P}(\mathbf{n})\cdot\mathbf{P}(-\mathbf{n}), \tag{41}$$

is just $C_2(\mathbf{P})$ up to a constant. The term involving $\frac{1}{J}$ requires some thought. The idea is to express $\frac{1}{J}$ as a power series around some finite value J_0 and then truncate the sum with N terms. Then we replace each of the integrals of $\frac{J}{J_0}$ by $C_p(\rho_N)$ up to constants. The natural value of J_0 is unity since for small displacements $\mathbf{Y}(\mathbf{r},t) = \mathbf{r} +$ small terms, and for $\mathbf{Y} = \mathbf{r}, J = 1$. For general J_0 we write

$$
\begin{aligned}
\frac{1}{J} &= \frac{1}{J_0 - (J_0 - J)} \\
&= \frac{1}{J_0}\sum_{k=0}^{\infty}(1 - \frac{J}{J_0})^k \\
&\approx \frac{1}{J_0}\sum_{k=0}^{N-1}(1 - \frac{J}{J_0})^k \\
&= \frac{1}{J}(1 - (1 - \frac{J}{J_0})^N),
\end{aligned} \tag{42}
$$

which is very nearly $\frac{1}{J}$ when $0 < J < 2J_)$ and N is large.

In the expression for the potential energy in H_N involving $\int d^2r\frac{1}{J}$ we make this replacement for J^{-1} and specifically set

$$\int d^2r J^{p+1} \longrightarrow \frac{(2\pi)^{2p+2}}{L^{2(2p+1)}} C_{p+1}(\rho_N), \tag{43}$$

so our truncated Hamiltonian is

$$H_N = \frac{1}{2}\sum_{\mathbf{n}=-M}^{M} \mathbf{P}(\mathbf{n}) \cdot \mathbf{P}(-\mathbf{n}) + \frac{g}{2J_0}\sum_{k=1}^{N} C_k^N \frac{(2\pi)^{2k-2}}{L^{2(2k-2)}} C_{k-1}(\rho_N). \tag{44}$$

This Hamiltonian, by construction, has zero Poisson bracket with $Q_N(\mathbf{n})$. Further $\{C_p(Q_N), H_N\} = 0$ as well.

This constitutes our mode truncated shallow water Hamiltonian and is an explicitly SU(N) symmetric approximation to the continuum shallow water theory from which we started. As the number of modes goes to infinity, the continuum theory is recovered in all its details. For finite N, the symmetry constraints of particle interchange are respected as accurately as possible.

5 Conclusions

In this talk we have presented insight into the origins of potential vorticity conservation and in doing so have investigated the algebra of infinitesimal operations associated with the particle interchange symmetry responsible for that conservation law. The generators of local infinitesimal particle interchanges are the potential vorticity at a point, and in the continuum theory their Poisson bracket algebra is infinite dimensional.

We then showed how to truncate the modes of the shallow water theory, expressed in Fourier space of its Lagrangian representation, and to alter the symmetry algebra so it remains a symmetry algebra of the finite degree of freedom theory. In the planar geometry where we worked, this replacement was straightforward.

In the future we shall address several questions:

- the application and numerical investigation of this kind of truncation to inviscid two dimensional incompressible flow. This simplest of all theories of fluid flow has only one Eulerian dynamical field which can be taken to be the vorticity out of the plane of flow, and the algebra of this variable in Eulerian representation parallels that discussed here for the potential vorticity.
- the extension of the construction presented here to two dimensional flow on a sphere (rotating, if you like).
- the extension of these ideas to planar and spherical stratified flow for the study of internal waves and surface waves.
- the numerical investigation of these symmetric finite degree of freedom systems to understand the role played by the symmetry constraints.
- investigation of the "statistical mechanics" of these symmetric Hamilton systems and of the paths to chaos in the systems.

Another avenue of substantial interest is to understand the Eulerian version of our Lagrangian formulations of these symmetric theories. This is both for general interest and since the numerical investigation of the symmetry preserving mode truncated theories may well be easier in Eulerian variables.

Finally, since damping and driving are physical ingredients of any real observations of the ocean, we expect that these inviscid or Hamiltonian discussions will serve as means for identifying variables in which to investigate both the inviscid and the dissipative physical settings. The advantage of the variables thus suggested is that when length scales and time scales are large enough that viscosity is unimportant, all conservation laws one would want to be respected are respected.

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