WHY EDDY DIFFUSIVITY DOESN'T WORK

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ABSTRACT

Three topics are discussed for which the concept of diffusion is inappropriate for summarizing the effect of small scale flows on larger flows. These three topics are lack of sufficient scale separation, conservation laws, and the mediating effect of intermediate scales to transfer non-diffusive effects from small scales to large.

INTRODUCTION

In modelling large scale flows in the ocean, one would like to be able to summarize the smaller scale processes in some simple way. A very common way is to use eddy viscosity, or other eddy diffusivities. The small-scale oceanographer is then supposed to supply the correct value of the diffusivity to be used in larger scales. The purpose of this paper is to point out that the use of eddy diffusivities is not an appropriate summary of small-scale phenomena, and a more complete description of these small features is required.

There are three topics that I will discuss to support this view. The first is that there is not a sufficient gap in scales between the small-scale flows one wishes to eliminate and the larger scale flows of interest. The second is the incompatibility of some conservation laws with diffusion. The third is that effects at small scales can be mediated by intermediate scales, leading to non-diffusive behavior.

NO SCALE GAP

We first discuss the absence of enough scale separation. It is true that if there were enough scale separation, the small scales would provide a diffusivity given by the Green-Kubo formula

$$D = \int_0^\infty \langle \mathbf{v}(t)\mathbf{v}(t+\tau)\rangle d\tau \qquad (1)$$

where $\mathbf{V}$ is the velocity followed by a Lagrangian tracer, (or, more generally, the velocity of the quantity being diffused), and the expression is written assuming $<\mathbf{V}> = 0$. The Green-Kubo formula is exactly what happens with thermal motions, providing the "molecular diffusivity". The thermal fluctuations have a wave number spectrum which is white

$$dE(\mathbf{k}) = \frac{k_B T}{(2\pi)^3} d^3\mathbf{k} \tag{2}$$

$$= \frac{k_B T}{(2\pi)^3} k^2 dk d^2\Omega$$

so that most of the velocity fluctuations are at very small scale. Ocean spectra, on the other hand, often have a negative power of k rather than a positive power. One cannot assume the scales are separated.

For molecular diffusivity, a cloud of tracer particles becomes diffuse as time goes on; the rms separation between two initially close particles is just $\sqrt{2}$ times the rms displacement of each one. The same value of the diffusivity applies to both the one-particle and two-particle cases. For "eddy diffusivity", this clearly doesn't work. The separation between particles is generally much smaller than $\sqrt{2}$ times the distance they have moved. This discrepancy is an indication that there is something wrong.

A way to understand the situation in which diffusivity does not work, is to use the approximation

$$D \approx \int_0^T <\mathbf{v}(t)\mathbf{v}(t+\tau)> d\tau \tag{3}$$

where T is the time scale on which the large-scale phenomena occur. (Thus D depends on more than the properties of the neglected small scales.) If the resulting value of D is not close to the infinite time limit, one should not expect diffusive behavior. Consider, for a moment, a small scale circular eddy. The Lagrangian velocity field is oscillating as the particle is carried around the eddy, and the time required for convergence of the Green-Kubo expression is likely to be very much larger than the rotation period of the eddy; if the convergence time is a year, the effect of the eddy cannot be summarized as diffusion for processes with characteristic times of a month. The typical sizes of flows with characteristic times short compared to the Green-Kubo convergence time can be considerably larger than the eddy under consideration.

It is the nature of diffusion that if there isn't any organized flow such as advection, the rms distance moved goes as the square root of the time. For non-diffusive cases, where eddy diffusion would often be used, various models have been worked out to give the time scaling of typical displacements. One of the earliest was Richardson (1926), who obtained $X \sim t^{3/2}$ for turbulent transport. This scaling has been refined, and other cases investigated by the Montroll school. Shlesinger and co-workers (1986), for example, have improved on Richardson's original work. More relevant to the ocean problems is the recent work by Young et al. (1989). They consider the transport of a tracer in a sequence of eddies considered to be stationary. Most of the tracer will stay at the outer edges of an eddy and be rapidly transported past the eddy. Some of the tracer, however, works its way into the eddy, where it is trapped for a significant time, and slowly released. If there were to be enough eddies, almost all of the tracer would eventually find itself in some eddy, and the diffusive regime would be recovered, but with a very small diffusivity determined by the trapping time. Most of the tracer would initially be transported significant distances i.e., several eddy diameters at a much higher rate than the ultimate asymptotic diffusion. The power law obtained by these workers depends on the specific model they make for the eddies. Young (1988) has also looked at the more general problem of transport with trapping.

CONSERVATION LAWS

The second topic concerns conservation laws. I will discuss this topic by means of an example. This example does not divide large scale from small scale, but divides submesoscale from even smaller scales. It concerns the interaction of submesoscale eddies with internal waves. The POLYMODE Local Dynamics Experiment group, McWilliams et al.((1983), find that an eddy viscosity estimated from a correlation between mesoscale horizontal shear and internal wave Reynolds stress implies a lifetime for a submesoscale eddy of 10 days, inconsistent with the observed much longer lifetime. (They speculate that absence of scale separation is responsible for the discrepancy; I will offer another explanation). This puzzle can be resolved by considering conservation laws. The integral of any function $F(\rho,J)$ of potential density ρ and potential vorticity J is conserved if dissipative processes are neglected. In fact, F is only advected by the water. Since internal waves propagate through the water, they carry zero value of each of these quantities. (Since J is nonlinear, the internal waves must include their nonlinear interactions in order to make this true.) Thus, interaction with internal waves can change the energy of an eddy, but cannot change the conservation of each F. If we assume the eddy is energetic relative to the internal waves with which it interacts, the second law of thermodynamics requires a random wave field to reduce the energy of the eddy. Eventually the eddy will have as little energy as it can, consistent with the conservation of each F.

A theorem known to Kelvin and proved by Arnol'd (1965) (quoted by Vallis et al., 1989) states that a steady flow is a stationary point of the energy with the constraint given by the conservation laws. This stationary point cannot be a local maximum, because a small amount of internal wave energy can always be added. It might be a saddle point, or it might be a minimum. Stability analysis suggests it is often a minimum. Stable solutions (neglecting the β effect) are circular vortices. The observed long-lifetime eddies do appear to be these circular solutions. The observations and the stability analysis are reviewed by McWilliams (1985).

If one attempted to describe the evolution of an eddy in a bath of internal waves using an effective diffusivity, the value of such a diffusivity would rapidly approach zero as the circular conditions were approached. (Nonetheless, the fluctuations that diffusion theory relates to dissipation would not diminish at all). As with the first topic I discussed, the effective diffusivity depends on the larger scale flow as well as on the eliminated smaller scales.

Vallis et al. (1989) look for extremal energy flows by evolving the equations of motion with a modification which dissipates energy but does not modify the vorticity invariants. Perhaps their modified equations (with the energy-dissipating parameter properly chosen) may model the actual large-scale flow in the presence of an internal-wave bath.

MEDIATION BY INTERMEDIATE SCALES

The third topic that illustrates flaws in the concept of eddy diffusivity involves even smaller scales. The large scales are now on the order of 10 meters, and the small scales are perhaps in the centimeter regime.

It is believed that the properties of the large scales in fully developed 3D turbulence are unaffected by changes in the viscosity. Doing anything to the small scales should, if eddy viscosity ideas apply, change the value of the eddy viscosity, but nothing more. In particular, with enough scale difference between the large scale flows and the small ones summarized by an eddy viscosity, the large scales should be unaffected.

But this is not what happens. It turns out that turbulent drag is reduced by adding very small amounts of long chain polymers to the water (or other fluid). These polymers directly influence rather small scales. I think the drag should be thought about as an energy loss, rather than as a force. The idea is that the interior of the flow dissipates energy, and the boundary layers adjust themselves to make the forces agree with the energetics. With this view, the drag has become inevitable when the largest scales of the flow, which are organized, break down into random flows at the next smaller scales. Doing something to the small scales has influenced the large scales.

The contrary view, widely held, is that the drag reduction is a boundary layer effect, and that the foregoing argument does not apply. Adherents of this view discount the experiments in which attempts were made to keep the polymers out of the boundary layer, and drag reduction still occurred (actually, it works even better).

As a result, experiments have been done avoiding boundaries completely. It appears that the spectrum of turbulence is steeper than without the polymers, with the large scales stronger and the small scales weaker. This is entirely consistent with my view that the energetics is dominant. In the drag reducing case, more energy remains in the larger scales, and less cascades to smaller scales.

This upscale influence must require the presence of the intermediate scales. It seems that the arguments which give rise to molecular diffusivities would apply in the absence of these scales. There is something, as yet not definitively identified, which works its way from scale to scale, upstream against the energy cascade, and changes the strength of that cascade. That something is not eddy viscosity, but something which knows more about the properties of the small scale flow.

CONCLUSION

The various topics I have treated show that the relevance of small-scale oceanography to large scales is more profound than can be summarized in a handful of parameters. Answers to questions discussed by small-scale modellers, such as whether small-scale flows are internal waves or vortical motions (Müller et al.,1988) have ramifications to larger scale questions such as the evolution of circular vortices.

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