HAMILTONIAN DESCRIPTION OF THE INTERACTION OF SURFACE WAVES WITH MIXED-LAYER CURRENTS

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ABSTRACT

The Hamiltonian for the dynamics of water motion with a free surface is constructed by combining the irrotational flow surface Hamiltonian and the interior rotational flow Hamiltonian. This Hamiltonian is then specialized to the effect of a depth-dependent current on small-amplitude surface waves. The dispersion relation and an associated variational principle are presented. The modification of the gravity-capillary group velocity by a wind-drift layer is remarkable.

INTRODUCTION

Ocean surface waves are an important driving mechanism for the mixed layer. Significant momentum and energy fluxes enter the mixed layer from the waves. In turn, the mixed layer acts back on the waves; its currents refract the waves, modifying their spectrum, and thereby modifying the energy and momentum fluxes to the mixed layer. The theory of Langmuir circulation is an outstanding example of such a feedback.

If the characteristic scales of the variation of the currents exceed a wavelength/4π in the vertical or about half a wavelength in the horizontal, the effect of the current on the wave is well described as a simple advection. (This is not to say that the consequences of advection are simple; Langmuir circulation could result from such advection.) It is often the case that this requirement is not met. The waves of interest may have wavelengths on the order of 4π times characteristic depths of currents. This paper describes an appropriate formulation for this case. This problem has been studied from other points of view; see Peregrine (1976) and Smith (1986).

There are a number of applications of the interactions of depth-dependent currents with waves. Not all of these involve what is usually thought of as mixed layer processes. Applications include Langmuir circulation, effects of currents on the evolution of the wave spectrum, wind-drift layer, wind-wave coupling, SAR observations of internal waves and other currents, and giant waves focussed by the ocean currents. All scales of waves are affected by depth-dependent currents, so these applications extend from centimeter to kilometer waves.

This paper does not discuss any application in detail, but is concerned with the general framework. It is becoming more widely appreciated that there are many advantages in a Hamiltonian description of the dynamics of fluid systems. We present a Hamiltonian for water motion involving both surface waves and rotational (vorticity-containing) subsurface flows.

THE HAMILTONIAN

The Hamiltonian is a function which allows equations of motion to be deduced. It exists whenever viscosity or other loss processes can be neglected. (Losses can be appended perturbatively, however.) Its value is the energy. The energy must be expressed in terms of canonical variables. An expression for the energy is usually readily available, so the harder part of providing a Hamiltonian is to identify the canonical variables. These variables come in conjugate pairs, generically named p_j , q_j. We have

$$H\left(p_j, q_j\right) = E \tag{1}$$

where H is the Hamiltonian and E is the energy. Hamilton's equations of motion follow from the canonical variational principle

$$\delta \int \left[\sum_j p_j \dot{q}_j - H\left(p_j, q_j\right) \right] dt = 0 \tag{2}$$

The statement that this principle is canonical means that p_j and q_j are to be varied independently, unlike Lagrange's variational principle which involves only the q_j's. The task of the Hamiltonian constructor is to find the p, q pairs and to express the energy in terms of them.

The Hamiltonian for pure surface waves, with only irrotational subsurface motion, is well known. It has been rediscovered several times. The first paper we are aware of is Zakharov (1968). Miles (1981) gives a review of the history. The canonical pairs are the surface elevation

$$q_j = \zeta(x, y) \tag{3}$$

and the velocity potential at the surface

$$p_j = \phi_s(x, y) = \phi\left(x, y, z = \zeta(x, y)\right) \tag{4}$$

The index j is to be thought of as a label for the point (x, y) giving the horizontal coordinates of the surface. The velocity is

$$\vec{v} = \nabla \phi \tag{5}$$

with

$$\nabla^2 \phi = 0 \tag{6}$$

And the energy (Hamiltonian) is

$$H[\zeta,\phi_s] = \int dx\, dy \left[\rho \int^{\zeta} \frac{v^2}{2} dz + \rho g\, \zeta^2/2 \right] \tag{7}$$

For now, we have left out surface tension. Later in the paper it will be included.

The Hamiltonian is also well known for rotational interior flow with no free boundaries. The canonical variables are Clebsch's potentials

$$p_j = \beta(x,y,z) \tag{8}$$

$$q_j = \alpha(x,y,z) \tag{9}$$

where j labels the point in space, and α, β are defined by the expression for the velocity

$$\vec{v} = \nabla\phi + \alpha\nabla\beta \tag{10}$$

The physical interpretation of this expression is described in Lamb (1932) p. 248. The potential ϕ is to be thought of as a functional of α, β given by

$$\nabla^2\phi + \nabla\cdot(\alpha\nabla\beta) = \nabla\cdot\vec{v} = 0 \tag{11}$$

or, alternatively, as a free variable without a conjugate partner, in the variational principle eq. (2), in which case eq. (11) follows. The energy is

$$H = \int dx\, dy\, dz\, \rho\, v^2/2 \tag{12}$$

In this paper we ignore stratification and the Earth's rotation. If we wished to include these effects, we would use Henyey's (1983) generalization of Clebsch's representation.

In order to construct the Hamiltonian for our problem, we must properly combine these two well-known theories. By experience, we know it is easy to write down incorrect combinations. We think that we should have been able to proceed deductively, but we actually found the correct combination by guesswork. Our result is a variational equation very similar to eq. (2), but it is not exactly a Hamiltonian in the usual sense, as described below. We slightly rewrite Clebsch's expression for the velocity by redefining ϕ (adding $\alpha\beta$ to the old ϕ) to get

$$\vec{v} = \nabla\phi - \beta\nabla\alpha \tag{13}$$

We use the step function at the surface

$$\theta = \theta(\zeta - z) = \begin{cases} 1 \text{ in the water} \\ 0 \text{ outside} \end{cases} \tag{14}$$

and its derivative, Dirac's delta function

$$\theta' = \delta(\zeta - z) \ . \tag{15}$$

The variational principle is

$$\delta \int \mathrm{L}\, dx\, dy\, dz\, dt = 0 \tag{16}$$

where

$$\mathrm{L} = \beta\dot{\alpha}\theta + \phi\dot{\zeta}\theta' - \frac{v^2}{2}\theta - g\,\frac{\zeta^2}{2}\theta' \tag{17}$$

The last two terms integrate to the energy. This is not exactly a canonical variational principle because the dynamical variable ζ appears in the θ and θ' of the first two terms.

As with the Clebsch case, ϕ in the interior can either be considered a functional of the other variables, or, more conveniently, as a free variable. The variable ζ is understood as independent of the the vertical coordinate z, while α, β, ϕ depend on all three coordinates. By variation with respect to all variables, we obtain the equations of motion

$$\dot{\alpha} + \vec{v}\cdot\nabla\alpha = 0 \tag{18}$$

$$\dot{\beta} + \vec{v}\cdot\nabla\beta = 0 \tag{19}$$

$$\nabla\cdot\vec{v} = 0 \tag{20}$$

$$\dot{\zeta} + \vec{v}_s\cdot\nabla\zeta = \vec{v}_s\cdot\hat{z} \tag{21}$$

$$\beta_s\,\dot{\alpha}_s - \dot{\phi}_s - \frac{v_s^2}{2} - g\,\zeta = 0 \tag{22}$$

The subscript s means evaluated on the surface; e.g., $\alpha_s(x,y) = \alpha(x,y,\zeta(x,y))$.

This set of equations is very general. The surface waves and the interior flow are arbitrarily nonlinear, and the relative scales are arbitrary. For many purposes (but not all), this generality only amounts to an unnecessary complication.

WAVES ON A CURRENT

Often, the horizontal scales of the currents are larger (a factor of two suffices) than those of the waves. In that case, short waves asymptotics can usually be used. To keep the discussion simple, and as the most important case, we assume that the surface waves can be treated linearly. We could handle moderate nonlinearities of the surface waves by themselves, using Whitham's method [Whitham (1974)].

For any linear wave system in the short-wavelength limit which has a Hamiltonian, there is a quadratic Hamiltonian such that

$$H = A\,\omega \tag{23}$$

where A is a conserved wave action and ω is the frequency. The wave momentum is given by the wave number

$$\vec{p} = A\vec{k} \tag{24}$$

The variational principle, in this limit, is

$$\delta \int dt \left[\vec{k}\cdot\dot{\vec{x}} - \omega(\vec{x},\vec{k}) \right] \tag{25}$$

where $\vec{x}$ is the center of action of a packet of the waves. The dispersion relation expresses the frequency in terms of the wave number and position. The position dependence arises, in our problem, by the (slow) horizontal dependence of the current. These results can be found, for example, in the work of Whitham (1974).

We are concerned with the case in which we cannot consider the currents as arbitrarily uniform with depth. Therefore, there does not exist a Lagrangian frame. We cannot, for this problem, use the restriction of Whitham's theory described by Bretherton and Garrett (1969). The frequency cannot be written as an intrinsic frequency depending only on wave number and a Doppler shift by a wave number-independent velocity. For the group velocity, we must use

$$\vec{v}_g = \partial\omega(\vec{x},\vec{k})/\partial\vec{k} \tag{26}$$

The dispersion relation follows from linearizing eqs. (18)–(22) about a state with a depth-dependent flow. The equations that result are not new. They are, for example, in Peregrine (1976). They are conveniently expressed in terms of a depth-dependent "intrinsic frequency"

$$\sigma(z) \equiv \omega - \vec{k}\cdot\vec{U}(z) \tag{27}$$

Then the excess of pressure over hydrostatic pressure obeys the equation

$$P''(z) - \frac{2\sigma'(z)}{\sigma(z)} P'(z) - k^2 P(z) = 0 \tag{28}$$

where

$$\sigma(0)P(0) = g\,P'(0)/\sigma(0) \tag{29}$$

and an appropriate boundary condition at the bottom; in deep water

$$P(-\infty) = 0 \tag{30}$$

Unless ω in eq. (27) is chosen properly, eqs. (28)–(30) have only the trivial solution $P = 0$. Thus, this system is very much like an eigenvalue problem. An approximate solution is

$$\omega \approx \sqrt{gk} + \vec{U}\left(z = 1/(2k)\right)\cdot\vec{k} \ , \tag{31}$$

with improvements [Smith (1986)] possible. For waves interacting with larger waves rather than currents, g is replaced by an effective g including the surface acceleration, and the advection $\vec{U}$ is evaluated exactly on the surface, not at a depth of $1/(2k)$ [Henyey et al. (1987)]. The equations of motion for wave packets involve all the partial derivatives of $\omega(\vec{x},\vec{k})$. Knowing only eqs. (28)–(30), and having the desire to do better than (31) or its improvements, calculating such derivatives requires solving the eigenvalue-like system at a set of values of $\vec{x},\vec{k}$, and taking differences.

Given the variational formulation, however, this procedure can be simplified. The general variational principle (16) induces a variational principle for the eigenvalue-like problem. As is familiar in eigenvalue problems (the "Hellmann-Feynman theorem"), this variational principle allows one to reduce the calculation of derivatives of ω to quadratures involving the unperturbed function (in our own case, the unperturbed pressure). The eigenvalue-like system need be solved only once at a $(\vec{k},\vec{x})$ point, and derivatives can be easily obtained.

One reduces the variational principle by ignoring horizontal dependence (due to the short-wavelength assumption) and Taylor-series expanding, retaining only quadratic terms (due to the linearity assumption). One assumes a fixed wave number and constrains the variables to be in the proper relationship as given by eqs. (18)–(22); in our problem we constrain all variables to be expressed in terms of the pressure. The pressure is related to the velocity potential by

$$P(z) = \sigma(z)\phi(z) \tag{32}$$

which is obtained by taking the time derivative of eq. (10), inserting eqs. (18)–(20), and comparing to Euler's equation. This is an application of a mathematical theorem, that if a variational problem is constrained by conditions true at an extremum, the extremum remains the same. On doing these manipulations and including surface tension γ, we find

$$0 = \delta \int dz \left[\frac{1}{2} \left[\frac{P'(z)}{\sigma(z)} \right]^2 + \frac{1}{2} \left[\frac{k\,P(z)}{\sigma(z)} \right]^2 - \frac{P(z)P'(z)}{g + \gamma k^2} \right] \tag{33}$$

The equivalences of this to eqs. (28)–(30) can easily be verified. We believe that this variational formulation is new.

In order to take derivatives, we abbreviate eq. (33) as

$$0 = \delta S / \delta P \tag{34}$$

For any solution P, another solution is a constant times P. The only way this can happen is

$$S = 0 \tag{35}$$

on the solution. Differentiating eq. (35) (with respect to any variable) we get

$$\int dz \left[\frac{\delta S}{\delta P(z)} d\,P(z) + \frac{\delta S}{\delta \sigma(z)} \left[d\,\omega - d\left(\vec{k} \cdot \vec{U}(z)\right) \right] \right] + \frac{\delta S}{\delta k} dk = 0 \tag{36}$$

By eq. (34), the first term is zero, so we do not have to evaluate $d\,P(z)$, which is hard. The remaining terms only involve easily evaluated explicit derivatives. We can solve for $d\,\omega$, the desired derivative. The result is

$$\vec{v}_g = \frac{\partial \omega}{\partial \vec{k}} = \frac{\int dz \left[\left\{ \frac{(P'(z))^2}{\sigma^3(z)} + \frac{(k\,P(z))^2}{\sigma^3(z)} \right\} \vec{U}(z) + \frac{\vec{k}P^2(z)}{\sigma^2(z)} + \frac{2\gamma \vec{k}}{(g + \gamma k^2)^2} P(z)P'(z) \right]}{\int \left[\frac{(P'(z))^2}{\sigma^3(z)} + \frac{(kP(z))^2}{\sigma^3(z)} \right] dz} \tag{37}$$

If there is a slow horizontal (x) variation in U, the spatial variation in frequency also follows from 36:

$$-\frac{dk_x}{dt} = \frac{\partial \omega}{\partial x} = \frac{\vec{k}\cdot\int \frac{\partial \vec{U}}{\partial x}(z,x)\left\{\frac{(P'(z))^2}{\sigma^3(z)} + \frac{(k\,P(z))^2}{\sigma^3(z)}\right\}dz}{\int\left\{\frac{(P'(z))^2}{\sigma^3(z)} + \frac{(k\,P(z))^2}{\sigma^3(z)}\right\}} \tag{38}$$

and the corresponding expression for $-dk_y/dt$.

WIND DRIFT LAYER

We have carried out the calculation outlined above for the case of a thin wind-drift layer. We assume $\vec{U}(z) = \hat{x}U_0 e^{-|z|/d}$. Since a wind drift layer is very thin, we have included capillarity in the dispersion relation. In figure 1 we compare the group velocity (for a wave in the $\hat{x}$ direction) for $U_0 =$ 40cm/s, $d =$ 8 mm with that of a uniform current of 40cm/s. Instead of the minimum at $\lambda =$ 4.4cm (Lamb, (1932) p. 460), there is a broad minimum at much larger wavelengths.

The resonant four-wave interaction, in the weak interaction limit, transfers energy at a rate proportional to $1/(\partial v_g/\partial k)$. We have not yet evaluated the four-wave coupling with the current present. If we assume that the wind-drift layer does not significantly reduce the coupling, then transfer rates will greatly increase with the layer present, since $\partial v_g/\partial k$ is so small. The non-linear transfer of energy in the moderately short wavelength part of the spectrum may have been grossly underestimated in the past by not including this effect.

SUMMARY

Problems involving the interaction between surface waves and currents can be usefully treated in the Hamiltonian framework. The Hamiltonian function is the energy expressed in terms of Clebsch's variables for the current, and the surfaces elevation and velocity potential for the surface waves. There is a variational principle associated with the Hamiltonian. Approximations and special cases are properly handled by using this variational principle. As an example, we have presented evidence that the four-wave resonant interaction is greatly enhanced for short waves due to the presence of a wind-drift layer.

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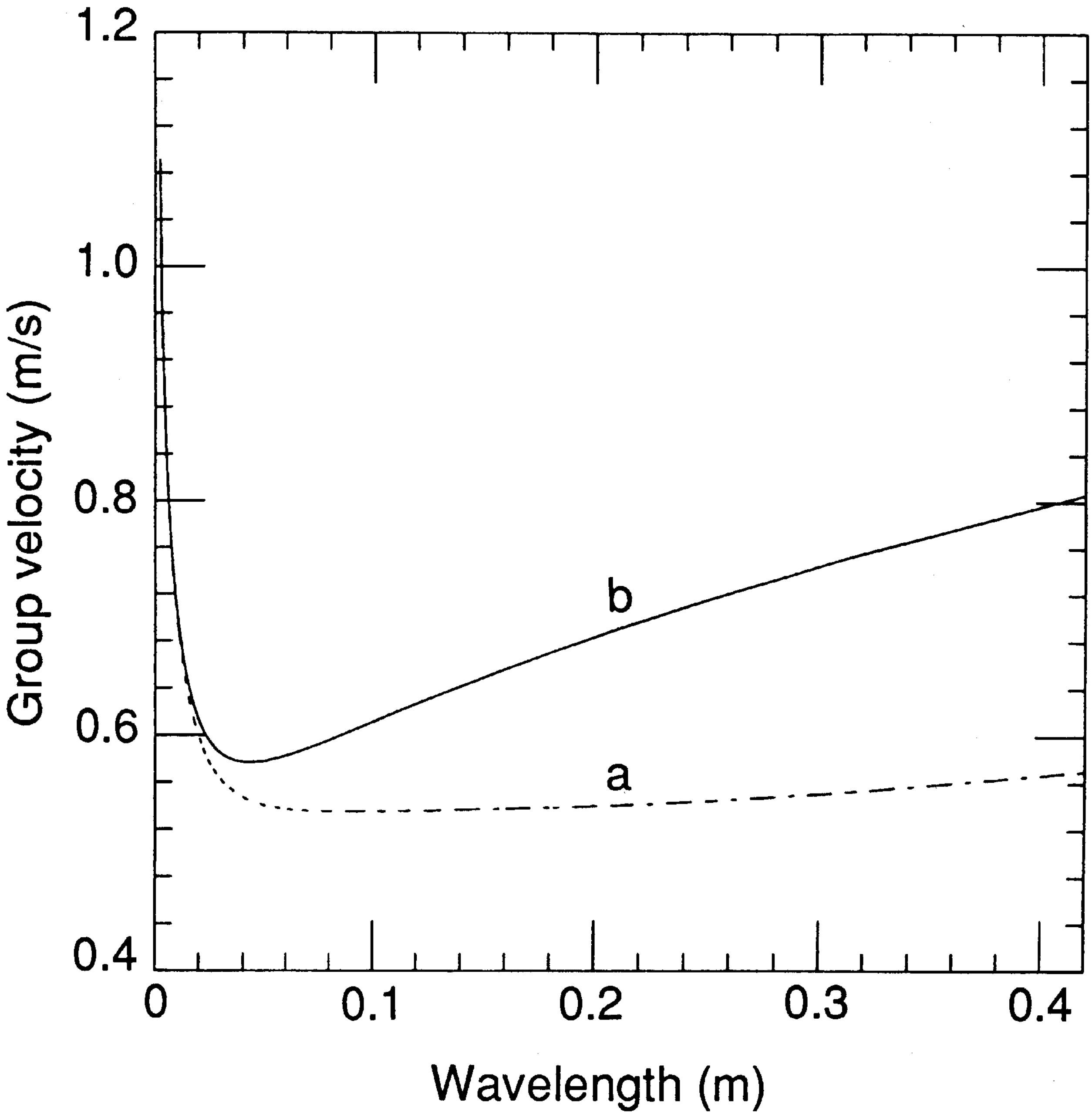

Fig. 1. Curve a is the group velocity for a wave moving in the direction of a current $U = U_0 e^{-|z|/d}$ with $U_0 = 0.4$m/sec and $d = 0.008$m. Curve b is the usual no current group velocity boosted by 0.4m/sec. The extremely flat curve for the group velocity should lead to considerably enhanced transport between different waves.

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