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On the asymmetry of baroclinic instability between easterly and westerly shear

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ABSTRACT

A complementary analysis of Charney-model baroclinic instability for a zonal current with easterly shear is presented. In contrast to an infinite number of unstable modes (the Charney, Green, and Burger modes) in westerly shearing currents, in easterly shearing flow, only a single unstable mode exists, corresponding to the Charney mode. For the same magnititude of vertical shear, the maximum growth rate in easterly shear is substantially smaller while the preferred wave length is considerably longer than the corresponding counterparts in westerly shear. The steering level of the most unstable mode is higher than one density scale height for easterly shear, while lower than one half density scale height for westerly shear. The asymmetry of baroclinic instability arises from the asymmetric variation of the basic state potential vorticity gradient with respect to the sign of the vertical shear due to the combined effects of the meridional variation of the Coriolis parameter and the vertical variation of density stratification.

1. Introduction

The environmental setting in which a monsoon depression or Africa wave is imbedded exhibits strong easterly shear (e.g., Mishra & Salvekar, 1980, Burpee, 1972). Disturbances developing in such an environment display an unstable baroclinic wave structure (e.g., Saha and Chang, 1982). This points to the necessity of clarifying the nature of baroclinic instability in an easterly shearing current. In the Eady (1949) model, the characteristics of baroclinic instability do not change when the vertical shear reverses its sign. This is simply an artifact due to the neglect of the meridional gradient of the basic state potential vorticity. Using a 2-layer β-plane model with unequal fluid depth, Pedlosky (1979) showed that the critical shear required for instability depends on the sign of vertical shear. In a more general framework, Moorthi and Arakawa (1985) demonstrated that baroclinic growth is asymmetric with respect to the sign of vertical shear due to the effect of non-zero Beta-effect and of vertical variation of the stratification. The purpose of this paper is to further elaborate upon this asymmetry by analyzing the stability of easterly currents using Charney's (1947) model. Several fundamental differences in baroclinic instability between easterly and westerly shears are emphasized.

2. Review of Charney model of zonal-flow instability

The basic zonal flow is $U(z)=\lambda_* z$ and the density is $\rho_s=\rho_0\exp(-z/H)$. The vertical shear λ_*, density scale height H, and Brunt-Vaisala frequency N are all assumed to be constant. Motion is confined in a semi-infinite domain on a β-plane channel with half width L. Let us scale horizontal coordinates x, y by L, horizontal velocity by $\lambda_* H$, time t by $L/\lambda_* H$, and ρ_s by ρ_0. The non-dimensional linear quasigeostrophic potential vorticity equation for adiabatic and frictionless motion is (Wang *et al.*, 1985, hereafter referred to as WBH)

$$\left(\frac{\partial}{\partial t}+z\frac{\partial}{\partial x}\right)\left[S\nabla^2\psi+\rho_s^{-1}\frac{\partial}{\partial z}\left(\rho_s\frac{\partial\psi}{\partial z}\right)\right]+\frac{\partial Q}{\partial y}\frac{\partial\lambda}{\partial x}=0. \quad (2.1)$$

The boundary conditions may be proposed as (Pedlosky, 1979):

$$\frac{\partial \psi}{\partial x}=0, \quad \text{at } y=\pm 1, \tag{2.2a}$$

$$\frac{\partial^2 \psi}{\partial t \partial z}-\frac{\partial \psi}{\partial x}=0, \quad \text{at } z=0, \tag{2.2b}$$

$$\int_{-1}^{1} \mathrm{d}y\, \rho_s \overline{\psi\left[\left(\frac{\partial}{\partial t}+z\frac{\partial}{\partial x}\right)\frac{\partial \psi}{\partial z}-\frac{\partial \psi}{\partial x}\right]} \rightarrow 0, \quad \text{as } z \rightarrow \infty. \tag{2.2c}$$

In eq. (2.1), ψ is the perturbation streamfunction, $S \equiv N^2H^2/f_0^2L^2$ is the Burger number and

$$\frac{\partial Q}{\partial y} \equiv 1+\frac{\beta N^2 H/f_0^2}{\lambda_*} \tag{2.3}$$

is proportional to meridional gradient of the basic state potential vorticity. Vertical shear λ_*, is positive for westerly shear and negative for easterly shear.

Following Charney and Stern (1962), a necessary condition for baroclinic instability can be obtained for the normal mode solution of the form $\psi=\mathrm{Re}\,\psi(z,y)\mathrm{e}^{ik(x-ct)}$, which is

$$\int_{-1}^{1}\mathrm{d}y\int_0^{\infty}\rho_s\frac{\partial Q/\partial y}{|z-c|^2}|\psi|^2\mathrm{d}z - \int_{-1}^{1}\mathrm{d}y\frac{|\psi|^2}{|c|^2}\bigg|_{z=0}=0. \tag{2.4}$$

For instability to exist in the Charney model, the meridional gradient of basic state potential vorticity, $\partial Q/\partial y$, must be positive. This can be satisfied for any value of westerly shear $\lambda_*>0$ or easterly shear greater than a critical value, i.e.,

$$\lambda_* < \lambda_{c*} = -\beta N^2 H/f_0^2. \tag{2.5}$$

Assuming $\psi(y,z)=\mathrm{e}^{z/2}\Phi(z)\cos ly$, where $l=(m+\frac{1}{2})\pi$, $m=0, 1, 2, \ldots$, defining a modified non-dimensional total horizontal wavenumber as

$$\tilde{K} \equiv (SK^2+\tfrac{1}{4})^{1/2} \tag{2.6a}$$

where $K^2=k^2+l^2$, and transforming vertical coordinates from z to

$$\xi=\tilde{K}(z-c), \tag{2.6b}$$

we formulate an eigenvalue problem for $\Phi(\xi)$ and eigenvalue $\sigma \equiv \tilde{K}c$:

$$\xi\frac{\mathrm{d}^2\Phi}{\mathrm{d}\xi^2}+(2\eta-\xi)\Phi=0, \tag{2.7}$$

$$\sigma\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}+\frac{1}{2K}\Phi\right)+\Phi=0, \quad \text{at } \xi=-\sigma; \tag{2.8a}$$

$\Phi(\xi)$ remains bounded as $\xi \rightarrow \infty$, (2.8b)

where

$$\eta \equiv \frac{\partial Q/\partial y}{2\tilde{K}}. \tag{2.9}$$

WBH have shown that the complex amplitude Φ is:

$$\Phi(\xi)=c_1(\eta)\Phi_1(\xi)+c_2(\eta)\Phi_2(\xi), \tag{2.10}$$

and the eigenvalue σ is the root of the dispersion equation

$$\sum_{n=0}^{N}A_n\sigma^n+\sigma\ln(-\sigma)\sum_{n=1}^{N-1}B_n\sigma^n=0, \quad N\rightarrow\infty. \tag{2.11}$$

In (2.10), the regular solution $\Phi_1(\xi)$ and singular solution $\Phi_2(\xi)$ are

$$\Phi_1(\xi)=\sum_{n=1}^{\infty}a_n\xi^n, \tag{2.12a}$$

$$\Phi_2(\xi)=\Phi_1(\xi)\ln\xi+\sum_{n=0}^{\infty}b_n\xi^n, \tag{2.12b}$$

where

$$a_1=1, \quad a_2=-\eta, \quad a_n=(a_{n-2}-2\eta a_{n-1})/[n(n-1)], \quad n\geq 3, \tag{2.13a}$$

$$b_0=-1/(2\eta), \quad b_1=0, \quad b_n=[b_{n-2}-2\eta b_{n-1}-(2n-1)a_n]/[n(n-1)], \quad n\geq 2, \tag{2.13b}$$

The coefficients are

$$c_2(\eta)=\frac{1}{2\pi i}(\mathrm{e}^{2i\pi\eta}-1), \tag{2.14a}$$

$$c_1(\eta)=c_2(\eta)\left[\ln 2-1+2\gamma+1/(2\eta)+\frac{\mathrm{d}}{\mathrm{d}\eta}\Gamma(1-\eta)\right], \tag{2.14b}$$

where $\gamma=0{\cdot}57721\cdots$ is the Euler constant, and $\Gamma(x)$ is the Gamma function. The logarithm is rendered single valued by choosing a branch cut such that

$$\ln \xi = \ln |\xi| + i \arg \xi, \quad -\pi < \arg \xi \leq \pi.$$

In eq. (2.11),

$$A_0 = c_2 b_0, \quad A_1 = c_2[(a_1 + b_0/(2\tilde{K})],$$

$$A_n = (-1)^{n+1}\{c_1[(n-1)a_n + a_{n-1}/(2\tilde{K})] + c_2[a_n + (n-1)b_n + b_{n-1}/(2\tilde{K})]\}, \quad n \geq 2, \tag{2.15}$$

$$B_n = (-1)^n c_2[na_{n+1} + a_n/(2\tilde{K})], \quad n \geq 1.$$

3. Stability of baroclinic easterly zonal currents

Numerical solution of the dispersion equation (2.11) requires truncation of the infinite series. We refer to eq. (2.11) as the Nth order truncation if the series is truncated at finite integer N. WBH have shown that for westerly shear, the third-order truncation yields satisfactory results. For easterly shear, however, the numerical solution becomes more cumbersome because of the slow convergence of the infinite series when η is small. The convergence of the numerical solution was tested for different truncation parameter N. When $N = 3$, eq. (2.11) has one westward propagating growing mode and a pair of complex conjugate roots. The growing mode is always identifiable and asymptotically converges to a limit as N increases (when λ_* changes to positive values, this mode recovers the Charney mode under westerly shear), whereas the other two roots are not traceable as N increases. The westward-propagating growing mode is a physical mode, while all other modes which are truncation-dependent are computational modes. The solution for the physical mode is found not to change when N exceeds 10; thus the 10th order truncation of eq. (2.11) was employed to calculate the growth rate and phase speed.

In Fig. 1, contours of $S^{1/2}Kc_i$, proportional to the growth rate, and of the phase speed c_r are displayed as functions of non-dimensional shear, $\lambda = \lambda_*/(\beta N^2 H/f_0^2)$, and wavenumber. For the convenience of comparison, the results for westerly shear $\lambda < 0$ are also plotted. In the region where $0 < \lambda < 0.2$, η is large, and $S_0^{1/2}Kc_i$ and c_r are too small to be plotted. The region in which $-1 < \lambda < 0$ represents a stable regime of the easterly shearing flow, in which the necessary condition for instability is not satisfied (see eq.(2.5)).

It should be pointed out that as $\lambda \to -1$ from below the parameter η tends to zero and progressively higher-order truncation is required in order to obtain an accurate solution. Thus, numerical computation becomes impossible. However, the solution near $\lambda = -1$ ($\eta = 0$) can be analytically derived. When $\lambda \to -1$, $\eta \to 0$, it follows from eq. (2.14) that $c_2(\eta) \to 0$ and $c_1(\eta) \to \frac{1}{2}$. From eq. (2.13a),

$$a_n = \begin{cases} 1/(2m+1)!, & n = 2m+1, \\ 0, & n = 2m, \end{cases} \quad m = 0, 1, 2, \ldots.$$

Thus, $c_1(\eta)\Phi_1(\xi) \to \frac{1}{2}\text{sh}\, \xi$, as $\eta \to 0$.

Likewise,

$$c_2(\eta)\Phi_2(\xi) \to \sum_{n=0}^{\infty} c_2(\eta) b_n \xi^n \to -\frac{1}{2}\text{ch}\, \xi, \quad \eta \to 0.$$

From eq.(2.10),

$$\lim_{\eta \to 0} \Phi(\xi) = \frac{1}{2}(\text{sh}\, \xi - \text{ch}\, \xi). \tag{3.1}$$

Substituting eq. (3.1) into eq. (2.8a) yields

$$\sigma = \frac{2\tilde{K}}{2\tilde{K} - 1},$$

or

$$c_i = 0, \quad c_r = \frac{1}{\tilde{K} - \frac{1}{2}}. \tag{3.2}$$

The numerical solution near $\lambda = -1$ indeed matches this analytical solution. Therefore, the unstable mode derived for $\eta \neq 0$ (when the critical layer exists) approaches the neutral solution at $\eta = 0$ (when the critical level disappears). The solution in the absence of the critical level comes from both the regular solution ($\frac{1}{2}$sh ξ) and the singular solution ($-\frac{1}{2}$ch ξ) without the logarithmic term involved. In fact, eq. (2.1) can be solved directly for $\eta = 0$, which gives the same result as eq. (3.2). If the Boussinesq approximation is further introduced, the solution reduces to $c_i = 0$, $c_r = 1/K$. The same neutral mode can be obtained in the Eady (1949) model (where $\eta = 0$,

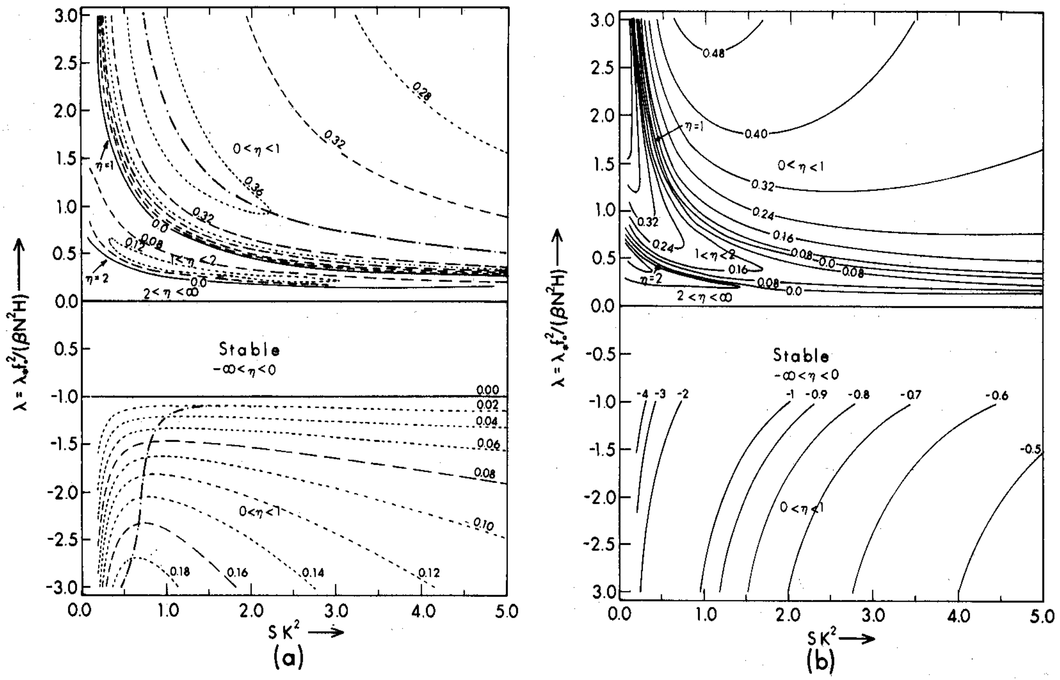

Fig. 1. Contour plot of (a) growth rate $S^{1/2}kc_i$ and (b) phase speed c_r for an inviscid non-Boussinesq fluid as a function of non-dimensional vertical shear λ and total wave-number squared SK^2. The dashed-dotted lines indicate the most unstable Charney mode.

i.e., the meridional gradient of the basic state potential vorticity vanishes) when the upper boundary tends to infinity, although the upper boundary condition is different.

The most striking feature shown in Fig. 1 is the asymmetry between westerly and easterly shear. Major differences may be summarized as follows.

(a) For westerly shearing current, $0 < \eta < \infty$, and in the wavenumber-vertical shear plane, there exists an infinite number of unstable regions which correspond to the Charney mode ($0 < \eta < 1$), Green (1960) mode ($1 < \eta < 2$), and Burger (1962) mode ($n < \eta < n+1$, $n = 2, 3, \cdots$), respectively. For an easterly shearing current, however, it follows from eqs. (2.9) and (2.3) that $-\infty < \eta < 1$; thus there is only a single unstable mode occurring for $0 < \eta < 1$ which corresponds to the Charney mode.

(b) For the same absolute value of non-dimensional vertical shear, the maximum growth rate in easterly shear is substantially less than that in westerly shear. The preferred wavelength of the most unstable mode in easterly shear is significantly longer than that in the westerly shear. The wave selection is clearer for the westerly shear than for the easterly shear, especially when $|\eta| < 2.0$.

(c) The propagation speed of the most unstable mode in easterly shear is generally much larger than that in westerly shear. The steering level where the local wind speed equals the phase speed of the most unstable wave is located in the lower troposphere for the westerly shearing current (lower than a half density scale height), while in the upper troposphere (higher than one density scale height) for the easterly shearing current.

4. Discussion

It is important to notice that the instability criterion eq. (2.5) indicates that the baroclinic instability of easterly shearing currents depends crucially on planetary rotational effect. Taking $H = 8 \times 10^3$ m and $N = 10^{-2}$ s^{-1}, at 45°N, with $f_0 = 10^{-4}$ s^{-1} and $\beta = 1.56 \times 10^{-11}$ m^{-1} s^{-1}, we find $\lambda_{c*} = -1.25$ ms^{-1} km^{-1}. Note that λ_{c*} decreases rapidly with decreasing latitude. For instance, at 20°N, where $f_0 = 0.5 \times 10^{-4}$ s^{-1} and $\beta = 2.13 \times 10^{-11}$ m^{-1} s^{-1}, λ_{c*} is equal to -6.82 ms^{-1} km^{-1}. This implies that baroclinic instability in subtropical easterly currents with constant shear can only rarely occur, since the instability requires an unusually large vertical easterly shear. This suggests that the horizontal shear and/or the vertical variation of the vertical shear associated with the internal jet structure may play important roles in generating dynamic instabilities in the subtropical regions where the easterlies prevail. One such example may be found for the Africa wave. Rennick (1976) showed that the mid-level easterly jet over North Africa is unstable to perturbations on the scale of Africa wave due to both of its horizontal and vertical shears. However, the quasi-geostrophic baroclinic energy conversion is not important in the generation of the waves, and the vertical transport of momentum which is proportional to the vertical shear alone does not act to support any instability. On the other hand, the horizontal momentum transport which is proportional to the horizontal shear alone still transfer energy from mean flow to the waves and support unstable waves.

Numerical calculations have previously been conducted by Moorthi and Arakawa (1985) using a vertical seven-level difference approximation of Charney model (hereafter referred as MA). Some differences between their results and the present result are noticeable, though there is a general qualitative agreement. First, in Fig. 2 and MA, short waves are stable with a cut-off wavelength of 2000 km for both easterly and westerly shear, while no short-wave cut-off exists in the continuous model. The short wave cut-off is an artifact of finite-difference approximation. Short waves have small vertical extent and thus cannot be resolved in a multi-level model. Second, although in both models, there exists long-wave cut-off for easterly shear, the cut-off wave-length in MA is only about 3500 km while in the continuous model it is about 15000–20000 km for $-4 < \lambda < 1.25$. Finally, for a similar basic state, the critical shear required for the instability of easterly shearing currents in the continuous model is nearly twice as large as that in the MA model. This implies that the asymmetry of the instability with respect to the sign of vertical shear is more prominent in the Charney model.

The asymmetric behavior of baroclinic instability results from the asymmetric variation of the basic state potential vorticity gradient with respect to the sign of vertical shear. In the Charney model, the dimensional basic state potential vorticity gradient is

$$\frac{\partial Q}{\partial y} = \beta - \frac{f_0^2}{N^2}\lambda_*\left(\rho_s^{-1}\frac{\partial \rho_s}{\partial z}\right).$$

In the presence of density stratification, the reverse of the sense of vertical shear causes asymmetric change in $\partial Q/\partial y$ if $\beta \neq 0$. Both the presence of the meridional variation of the Coriolis parameter and the vertical variation of density stratification are necessary to create the asymmetry.

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