LECTURE 5: EARTH’S FIGURE AND GRAVITY

The figure of the earth is a smooth closed surface that represents the shape of a gravitational equipotential surface (sea level). It is a surface upon which more complicated topography can be represented.

The International Reference Ellipsoid is an ellipsoid with dimensions:

- Equatorial Radius: \( a = 6378.136 \text{ km} \)
- Polar Radius: \( c = 6356.751 \text{ km} \)
- Radius of Equivalent Sphere: \( R = 6371.000 \text{ km} \)
- Flattening: \( f = 1/298.252 \)
- Acceleration Ratio: \( m = \frac{a_c}{a_G} = \frac{\omega^2 a^3}{GM_E} = 1/288.901 \)
- Moment of Inertia Ratio: \( H = \frac{C - A}{C} = 1/305.457 \)

The gravitational potential of a body is given by MacCullagh’s formula:

\[
U_G = -G \frac{M_E}{r} - G \frac{(A + B + C - 3I)}{2r^3}
\]

where \( A, B, \) & \( C \) are the moments of inertia about the \( x, y \), & \( z \) axes, and \( I \) is the moment about the OP axis.

If \( A=B \), then:

\[
I = A \sin^2 \theta + C \cos^2 \theta
\]

\[
U_G = -G \frac{M_E}{r} - G \frac{(C - A)}{r^3} \left( 3 \cos^2 \theta - 1 \right) = -G \frac{M_E}{r} - G \frac{(C - A)}{r^3} P_2(\cos \theta)
\]

More generally:

\[
U_G = -G \frac{M_E}{r} \left( 1 - \sum_{n=2}^{\infty} \left( \frac{R}{r} \right)^2 J_n P_n(\cos \theta) \right)
\]

Where \( P_n \) are the Legendre polynomials and the coefficients \( J_n \) are measured for Earth. The most important is the dynamical form factor: \( J_2 = \frac{C - A}{M_E R^2} = 1082.6 \times 10^{-6} \).
The next term, $J_3$, describes pear-shaped variations: a $\sim 17$ m bulge at North pole and $\sim 7$ m bulges at mid-southern latitudes ($\sim 1000$ times smaller than $J_2$)

The gravitational potential of the Earth (the geopotential) is given by:

$$U_g = U_0 - \frac{1}{2} \omega^2 r^2 \sin^2 \theta$$

$$U_g = -\frac{GM}{r^3} (C - A) \left( \frac{3 \cos^2 \theta - 1}{2} \right) - \frac{1}{2} \omega^2 r^2 \sin^2 \theta$$

The geopotential is a constant ($U_0$) everywhere on the reference ellipsoid. Then:

At the equator:  
$$U_0 = -\frac{GM}{a} + \frac{G}{2a^3} (C - A) - \frac{1}{2} \omega^2 a^2$$

At the pole:  
$$U_0 = -\frac{GM}{c} + \frac{G}{c^3} (C - A)$$

Then:

$$f = \frac{a - c}{c} = \frac{(C - A)(a^2 + 2c)}{M_E a^2} + \frac{1}{2} \frac{a^2 \omega^2}{GM_E} = \frac{3}{2} J_2 + \frac{1}{2} m$$

Where we have approximated $a \approx c$ on the right hand side.

This allows us to write:  
$$J_2 = \frac{1}{3} (2f - m) = 0.33 \left( \frac{1}{300} \right)$$

But  
$$J_2 = \frac{C - A}{M_E R^2} = \frac{C}{M_E R^2} H = \frac{C}{M_E R^2} \left( \frac{1}{300} \right)$$

This indicates that $C = 0.33 M_E R^2$ which is smaller than the $C = 0.4 M_E R^2$ expected for a sphere of uniform density. Thus, the earth gets denser closer to its center.

**The reference ellipsoid:**

To first order:  
$$r = a \left(1 - f \sin^2 \lambda \right)$$

Geocentric latitude = $\lambda$  
(measured from center of mass)

Geographic latitude = $\lambda_g$ (in common use)

To first order:  
$$\sin^2 \lambda = \sin^2 \lambda_g - f \sin^2 2\lambda_g$$
The acceleration of gravity on the reference ellipsoid is given by: \( \ddot{g} = -\ddot{r}U_g \)

Performing this differentiation gives:

\[
\begin{align*}
|g| &= \frac{GM}{r^2} - \frac{3GM_Ea^2J_2}{r^2} \frac{3\sin^2\lambda - 1}{2} - \omega^2 r \cos^2\lambda \\
\end{align*}
\]

Rewriting gives:

\[
|g| = \frac{GM}{a^2} \left( 1 + 2f \sin^2\lambda \right) - 3J_2 \left( \frac{3\sin^2\lambda - 1}{2} \right) - m\left( 1 - \sin^2\lambda \right)
\]

Equatorial gravity is then:

\[
g_e = \frac{GM}{a^2} \left[ 1 - \frac{3}{2}J_2 - m \right] = 9.780327 \text{ m/s}^2
\]

To first order, the variation of gravity can then be written as:

\[
g = g_e \left[ 1 + \left( 2m - \frac{3}{2}J_2 \right) \sin^2\lambda \right]
\]

Writing in terms of \( \lambda_g \) gives:

\[
g = g_e \left[ 1 + \left( \frac{3}{2}f - \frac{17}{14}mf \right) \sin^2\lambda_g + \left( \frac{f^2}{8} - \frac{5}{8}mf \right) \sin^2 2\lambda_g \right]
\]

\[
g = 9.780327 \left[ 1 + 0.0053024 \sin^2\lambda_g + 0.0000059 \sin^2 2\lambda_g \right]
\]

Setting \( \lambda_g = 90^\circ \) and dropping higher order terms yields:

\[
\frac{g_p - g_e}{g_e} = \frac{5}{2} \left[ 2m - f \right]
\]

This allows us to compute the polar gravity: \( g_p = 9.832186 \text{ m/s}^2 \)

The poleward increase in gravity is 5186 mgal, and thus only about 0.5% of the absolute value (gravity is typically measured in units of mgal = \( 10^{-5} \text{ m/s}^2 \)).

Gravity decreases toward the pole because the pole:

(1) is closer to the center of Earth than the equator (6600 mgal)

(2) does not experience centrifugal acceleration (3375 mgal)

But the equator has more mass (because of the bulge), which increases the equatorial gravity. Together these three affect yield the 5186 mgal difference.