Analytical Groundwater Flow Solutions

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Steady One-Dimensional Flow

For ground water flow in the x-direction in a confined aquifer:
\[ \frac{dh}{dx} = 0 \]
Integrate twice:
\[ h = Cx + h_0 \]
\[ dh/dx = -q/K, \text{ according to Darcy's law (} q = Q/A) \]
This states that head varies linearly with flow in the x-direction.

Response of ideal aquifers to pumping

- Assumptions:
  - Governing equation:
    - compressibility is strictly vertical
    - water release is instantaneous as head drops
    - vertically integrated flow equation (vertical gradients are negligible)
Response of ideal aquifers to pumping
• Aquifer characteristics
  ■ homogeneous and isotropic aquifers
  ■ constant thickness
  ■ hydraulic head is uniform prior to pumping
  ■ aquifer is horizontal and infinitely large in the horizontal direction
• Well and pumping characteristics
  ■ single, fully penetrating well pumping at a fixed rate
  ■ well diameter is infinitesimally small

Steady Radial Flow to a Well-Confined
For horizontal flow, Q at any radius r, from Darcy’s law,

\[ Q = -2\pi rbK \frac{dh}{dr} \]

for steady radial flow to a well where Q, b, K are const

Steady Radial Flow to a Well-Confined
Integrating after separation of variables, with \( h = h_w \) at \( r = r_w \) at the well, yields Thiem Eqn

\[ Q = 2\pi Kb\left(\frac{h-h_w}{(\ln(r/r_w))}\right) \]
Steady Radial Flow to a Well-
Confined

- Near the well, transmissivity, \( T \), may be estimated by observing heads \( h_1 \) and \( h_2 \) at two adjacent observation wells located at \( r_1 \) and \( r_2 \), respectively, from the pumping well

\[
T = Kb = \frac{Q \ln(r_2 / r_1)}{2\pi(h_2 - h_1)}
\]

Steady Radial Flow to a Well-
Unconfined

- Using Dupuit’s assumptions and applying Darcy’s law for radial flow in an unconfined, homogeneous, isotropic, and horizontal aquifer yields:

\[
Q = -2\pi K h \frac{dh}{dr}
\]

integrating,

\[
Q = \pi K (h_2^2 - h_1^2) / \ln(r_2 / r_1)
\]

solving for \( K \),

\[
K = \frac{Q}{\pi(h_2^2 - h_1^2)} \ln \left( \frac{r_2}{r_1} \right)
\]

\( h_1 \) and \( h_2 \) are observed at adjacent wells, distances \( r_1 \) and \( r_2 \) from the pumping well.
Transient condition: The Theis solution

Two-dimensional groundwater flow in a confined aquifer with transmissivity $T$ and storativity $S$:

$$
\frac{\partial}{\partial x} \left( \frac{K_x}{h} \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{K_y}{h} \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{K_z}{h} \frac{\partial h}{\partial z} \right) = S \frac{\partial h}{\partial t}
$$

Saturated 3-D equation

Can be written as:

$$
\frac{\partial h}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial h}{\partial r} \right) + \frac{\partial^2 h}{\partial \theta^2} + \frac{\partial^2 h}{\partial \phi^2} = \frac{S}{T} \frac{\partial h}{\partial t}
$$

Initial condition:

$h(r,0) = h_0$ for all $r$

Boundary condition at $r=\infty$:

$h(r,t) = h_0$ for all $t$

At the well face (Darcy’s law):

$$
\frac{\partial h}{\partial r} = \frac{Q}{2\pi T r} \quad \text{for} \quad r > 0
$$
Transient condition: The Theis solution

\[ s = \frac{Q}{4\pi T} \int_0^r \left( \frac{1}{2} - \frac{1}{6} \frac{t}{r^2} - \frac{1}{12} \frac{t^2}{r^4} \right) \, dr \]

\( s \) = drawdown = \( h_0 - h \); \( h_0 \) = initial head; \( h \) = head at time \( t \); \( Q \) = pumping rate; \( T \) = transmissivity

\( \mu = \frac{S\pi}{4T} \)

\( S \) = storativity (storage coefficient); \( r \) = distance from the well; \( t \) = time; \( W(\mu) \) = well function

Multiple-Well Systems
Impermeable boundary

Perennial stream

Injection-Pumping Pair of Wells
When pumping starts from a well in a leaky aquifer, drawdown of the piezometric surface can be given by:

\[ s = \frac{Q}{4\pi T} W(u, r/B) \]

\[ r/B = r/ T/(K' / b') \]

- \( T \) = transmissivity of the aquifer
- \( K' \) = vertical hydraulic conductivity
- \( b' \) = thickness of the aquitard
Other well analytical solutions

- Partially penetrating well
- Two or 3 layer system
- Large diameter well
- Unsaturated/saturated condition